## Online Appendix to Robust Intervention in Networks

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## Abstract

We show that Property B holds if and only if  $(\psi^0 + \psi) \notin Z$ . We also provide additional sufficient conditions under which Property B holds, along with a graphical example of the two-agent case.

## **1** Proof of Equivalence

**Property B** in the main text requires that the solution to the DM's robust optimization problem contains no zero entries (non-negligence condition).

Recall that the objective function  $f : \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}$  is defined as

$$f(\mathbf{x}, \mathbf{B}) = \frac{1}{2} \left( \langle \mathbf{x}, \mathbf{M} \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{B} \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{C} \mathbf{x} \rangle - 2 \langle \varphi, \mathbf{x} \rangle + \underbrace{||\mathbf{z}||^2 + ||\mathbf{C}^{\frac{1}{2}} \mathbf{x}^0||^2}_{\text{constant}} \right)$$

where  $\varphi = \psi^0 + \psi$  with  $\psi^0 = \mathbf{C}\mathbf{x}^0$  and  $\psi = \sum_{i=1}^n z_i \mathbf{m}_i$ . Define  $g_i(\mathbf{x}) = \max_{\mathbf{B}_i \in \mathcal{B}_i} \frac{1}{2} \langle \mathbf{x}, \mathbf{B}_i \mathbf{x} \rangle$  for each  $i \in N$ , which is convex and has its minimum at **0**. Let  $g(\mathbf{x}) = \sum_{i=1}^n g_i(\mathbf{x})$ . Let  $\mathbf{x}^*$  be defined as

$$\mathbf{x}^* = \operatorname*{arg\,min}_{\mathbf{x} \in \mathbb{R}^n} \max_{\mathbf{B} \in \mathcal{B}} f(\mathbf{x}, \mathbf{B}) = \operatorname*{arg\,min}_{\mathbf{x} \in \mathbb{R}^n} \left[ g(\mathbf{x}) + \frac{1}{2} \langle \mathbf{x}, \mathbf{D} \mathbf{x} \rangle - \langle \varphi, \mathbf{x} \rangle + \operatorname{constant} \right],$$

where  $\mathbf{D} = \mathbf{M} + \mathbf{C}$ .  $\mathbf{x}^*$  is determined by the first-order condition,  $\varphi \in \mathbf{D}\mathbf{x}^* + \partial g(\mathbf{x}^*)$ , where  $\partial g(\mathbf{x})$  denotes the set of subgradients of g at  $\mathbf{x}$ , defined by

$$\partial g(\mathbf{x}) = \{ \mathbf{u} \in \mathbb{R}^n \, | \, g(\mathbf{y}) - g(\mathbf{x}) \ge \langle \mathbf{u}, \mathbf{y} - \mathbf{x} \rangle \text{ for all } \mathbf{y} \in \mathbb{R}^n \}.$$

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The first order condition implies that  $\mathbf{x}^*$  has a zero entry if and only if

$$\varphi \in Z = \{ \mathbf{D}\mathbf{x} + \partial g(\mathbf{x}) \, | \, \mathbf{x} \text{ has a zero entry} \}. \tag{1}$$

Consequently, Property B in the main text is equivalent to the condition  $\varphi \notin Z$ .

## 2 Sufficient Conditions with An Example

Roughly speaking, Property B requires that Z is small in  $\mathbb{R}^n$ . Z is small if and only if  $\max_{i,j} \mathbf{v}_{ij}^2$  is relatively smaller than the eigenvalues of  $\mathbf{D} = \mathbf{M} + \mathbf{C}$ . To illustrate these points, consider a network of two agents. Then, for each *i*, the subgradient is calculated as

$$\partial g_i(\mathbf{x}) = \begin{cases} \{x_1(\mathbf{v}_{i1}^2, t\mathbf{v}_{i1}\mathbf{v}_{i2})^\mathsf{T} \mid t \in [-1, 1]\} & \text{if } x_1 \neq 0, x_2 = 0, \\ \{x_2(t\mathbf{v}_{i1}\mathbf{v}_{i2}, \mathbf{v}_{i2}^2)^\mathsf{T} \mid t \in [-1, 1]\} & \text{if } x_1 = 0, x_2 \neq 0. \end{cases}$$

Figure 1 illustrates Z under the assumption of  $\mathbf{M} + \mathbf{C} = k\mathbf{I}$  for some k > 0 for simplicity. In the left figure, the set { $\mathbf{x}$  has a zero entry} is represented as the union of two colored lines. The blue line represents the set of  $\mathbf{x}$  with  $x_1 = 0$ , and the red line represents the set of  $\mathbf{x}$  with  $x_2 = 0$ . Consider a set-valued function  $\mathbf{D} + \partial g : \mathbb{R}^2 \to \mathbb{R}^2$ defined as  $(\mathbf{D} + \partial g)(\mathbf{x}) = \mathbf{D}(\mathbf{x}) + \partial g(\mathbf{x})$ . In the right figure in Figure 1, the blue region is the image of the blue line { $\mathbf{x} \in \mathbb{R} | x_1 = 0$ }, and the red region is the image of the red line { $\mathbf{x} \in \mathbb{R} | x_2 = 0$ }. Property B requires that  $\varphi$  is not contained in either of the two regions in the right figure. Since k > 0, the union of the two regions is not only strictly contained in  $\mathbb{R}^2$ , but also strictly decreases in size as k increases. The union of the two regions is closed. Moreover, the measure of the union is nonzero. Consequently, Property B does not hold generically, and the interior of the complement of Z is not empty.



Figure 1: Illustration of Z as an image of  $\mathbf{D}\mathbf{x} + \partial g(\mathbf{x})$ 

We now explain how the size of Z is determined. Without loss of generality, let us consider the size of the red region in the right figure. For a given value of  $x_1$  on the horizontal axis in the left figure, the subgradient is a vertical segment in the red region in the right figure. For example, at  $\mathbf{x} = (1,0)$ ,  $\mathbf{D}\mathbf{x} + \partial g(\mathbf{x}) = \{(k + \mathbf{v}_{i1}^2, t\mathbf{v}_{i1}\mathbf{v}_{i2})^{\mathsf{T}} | t \in [-1,1]\}$ . The height of the segment is determined by the values of  $\mathbf{v}_{i1}$  and  $\mathbf{v}_{i2}$ . Thus, when other parameters are equal, **Property B** holds if the size of the variances becomes sufficiently small. Similarly, for given values of the variance, Property *B* holds if *k* is sufficiently large. Furthermore, it follows that for a given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $\frac{\max_{i,j} \mathbf{v}_{ij}^2}{\text{smallest eigenvalue of } \mathbf{D}} < \delta$ , then there exists a modified target  $\tilde{\mathbf{z}}$  of  $\mathbf{z}$  such that  $|\tilde{\mathbf{z}} - \mathbf{z}| < \varepsilon$ , and the corresponding optimal intervention  $\mathbf{x}^*$  with respect to the new target  $\tilde{\mathbf{z}}$  has no zero entry.