A HODGE THEORETIC EXTENSION OF SHAPLEY AXIOMS

TONGSEOK LIM

Abstract. Lloyd S. Shapley [19, 20] introduced a set of axioms in 1953, now called the *Shapley axioms*, and showed that the axioms characterize a natural allocation among the players who are in grand coalition of a *cooperative game*. Recently, Stern and Tettenhorst [23] showed that a cooperative game can be decomposed into a sum of *component games*, one for each player, whose value at the grand coalition coincides with the *Shapley value*. The component games are defined by the solutions to the naturally defined system of least squares linear equations via the framework of the *Hodge decomposition* on the hypercube graph.

In this paper we propose a new set of axioms which characterizes the component games. Furthermore, we realize them through an intriguing stochastic path integral driven by a canonical Markov chain. The integrals are natural representation for the expected total contribution made by the players for each coalition, and hence can be viewed as their fair share. This allows us to interpret the component game values for each coalition also as a valid measure of fair allocation among the players in the coalition. Our axioms may be viewed as a completion of Shapley axioms in view of this characterization of the Hodge-theoretic component games, and moreover, the stochastic path integral representation of the component games may be viewed as an extension of the *Shapley formula*.

Keywords: Shapley axioms, Shapley value, Shapley formula, cooperative game, component game, Hodge decomposition, least squares, path integral representation
MSC2020 Classification: 91A12, 05C57, 68R01

Tongseok Lim: Krannert School of Management
Purdue University, West Lafayette, Indiana 47907, USA
E-mail address: lim336@purdue.edu.
Date: September 13, 2021.

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers. For $N \in \mathbb{N}$, let $[N] := \{1, 2, ..., N\}$ denote the set of players. The set of *cooperative games* is defined by

$$\mathcal{G}_N = \{v : 2^{[N]} \to \mathbb{R} \mid v(\emptyset) = 0\}.$$

Thus a cooperative game here is simply a (value) function on the subsets of $[N]$, where each $S \subseteq [N]$ represents a coalition of players in S, and $v(S)$ represents the value assigned to the coalition S, where the null coalition $\emptyset$ receives zero value.

Lloyd Shapley considered the question of how to split the grand coalition value $v([N])$ among the players for a given game $v \in \mathcal{G}_N$. It is uniquely defined according to the following theorem, which is quoted from Stern and Tettenhorst [23].

Theorem 1.1 (Shapley [20])**.** *There exists a unique allocation* $v \in \mathcal{G}_N \mapsto \big(\phi_i(v)\big)_{i\in[N]}$ *satisfying the following conditions:*

- efficiency: $\sum_{i\in[N]} \phi_i(v) = v([N])$.
- symmetry: $v\big(S \cup \{i\}\big) = v\big(S \cup \{j\}\big)$ *for all* $S \subseteq [N] \setminus \{i, j\}$ *yields* $\phi_i(v) = \phi_j(v)$.
- null-player: $v\big(S \cup \{i\}\big) - v(S) = 0$ *for all* $S \subseteq [N] \setminus \{i\}$ *yields* $\phi_i(v) = 0$.
- linearity: $\phi_i(\alpha v + \alpha' v') = \alpha\phi_i(v) + \alpha'\phi_i(v')$ *for all* $\alpha, \alpha' \in \mathbb{R}$ *and* $v, v' \in \mathcal{G}_N$.

Moreover, this allocation is given by the following explicit formula:

$$\phi_i(v) = \sum_{S\subseteq[N]\setminus\{i\}} \frac{|S|!\big(N - 1 - |S|\big)!}{N!}\Big(v\big(S \cup \{i\}\big) - v(S)\Big). \tag{1.1}$$

The four conditions listed above are often called the *Shapley axioms*. Quoted from [23], they say that [efficiency] the value obtained by the grand coalition is fully distributed among the players, [symmetry] equivalent players receive equal amounts, [null-player] a player who contributes no marginal value to any coalition receives nothing, and [linearity] the allocation is linear in the game values.

(1.1) can be rewritten also quoted from [23]: Suppose the players form the grand coalition by joining, one-at-a-time, in the order defined by a permutation σ of $[N]$. That is, player i joins immediately after the coalition $S_{\sigma,i} = \big\{j \in [N] : \sigma(j) < \sigma(i)\big\}$ has formed, contributing marginal value $v\big(S_{\sigma,i} \cup \{i\}\big) - v(S_{\sigma,i})$. Then $\phi_i(v)$ is the average marginal value contributed by player i over all $N!$ permutations σ, i.e.,

$$\phi_i(v) = \frac{1}{N!} \sum_{\sigma} \Big(v\big(S_{\sigma,i} \cup \{i\}\big) - v(S_{\sigma,i}) \Big). \tag{1.2}$$

We see an important principle here, which we may call the *Shapley's principle*:

Shapley's principle. The value allocated to player i is based entirely on the marginal values $v\big(S \cup \{i\}\big) - v(S)$ the player i contribute.

Extensive research on the cooperative and noncooperative games have been inspired by and evolved from the pioneering study by Shapley [19–22], and various concepts of solutions have been proposed, e.g., Kalai and Samet [8], Ruiz et al. [18] and Kultti and Salonen [12]. In particular, the combinatorial Hodge decomposition has recently been applied to game theory in various contexts, including noncooperative games (Candogan et al. [1]), cooperative games (Stern and Tettenhorst [23]), and also other interesting problems in economics, e.g., ranking of social preferences (Jiang et al. [7]). In addition, there have been efforts to model various cooperation restrictions, e.g., Faigle and Kern [5], Khmelnitskaya et al. [9], and Koshevoy et al. [11]. On the other hand, computational aspects of Shapley theory have been studied by Castro et al. [2, 3] and Deng and Papadimitriou [4]. We refer to Lim [13] for an elementary introduction to the Hodge theory on graphs, while more general and profound theory of Hodge goes back to Hodge [6] and Kodaira [10]. For more comprehensive treatment of game theory, we refer to e.g. Roth [16, 17], Peters [15].

In particular, Stern and Tettenhorst [23] showed that, given a game $v \in \mathcal{G}_N$, there exist *component games* $v_i \in \mathcal{G}_N$ for each player $i \in [N]$ which are naturally defined via the *combinatorial Hodge decomposition*, satisfying $v = \sum_{i \in [N]} v_i$. Moreover, it holds $v_i([N]) = \phi_i(v)$, hence they obtained a new characterization of the Shapley value as the value of the grand coalition in each player's component game.

In this context, the *combinatorial Hodge decomposition* boils down to the *Fundamental Theorem of Linear Algebra*. For finite-dimensional inner product spaces X, Y and a linear map $\mathrm{d} : X \to Y$ and its adjoint $\mathrm{d}^* : Y \to X$ given by $\langle \mathrm{d}x, y \rangle_Y = \langle x, \mathrm{d}^* y \rangle_X$, FTLA asserts that the orthogonal decompositions hold:

$$X = \mathcal{R}(\mathrm{d}^*) \oplus \mathcal{N}(\mathrm{d}), \qquad Y = \mathcal{R}(\mathrm{d}) \oplus \mathcal{N}(\mathrm{d}^*),$$

where $\mathcal{R}(\cdot)$, $\mathcal{N}(\cdot)$ stand for the range and nullspace respectively.

In order to introduce the work of [23], let us briefly review the setup. Let $G = (V, E)$ be an oriented graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. "Oriented" means at most one of (a, b) and (b, a) is in E for $a, b \in V$. If $f \colon E \to \mathbb{R}$ and $(a, b) \in E$, define $f(b, a) \coloneqq -f(a, b)$ for the reverse-oriented edge. Let $\ell^2(V)$ be the space of functions $V \to \mathbb{R}$ with the inner product

$$\langle u, v \rangle \coloneqq \sum_{a \in V} u(a)v(a).$$

Similarly, denote by $\ell^2(E)$ the space of functions $E \to \mathbb{R}$ with inner product

$$\langle f, g \rangle \coloneqq \sum_{(a,b) \in E} f(a, b)g(a, b).$$

Next, define a linear operator $\mathrm{d} \colon \ell^2(V) \to \ell^2(E)$, the *gradient*, by

$$\mathrm{d}u(a, b) \coloneqq u(b) - u(a).$$

Then its adjoint $\mathrm{d}^* \colon \ell^2(E) \to \ell^2(V)$, the (negative) *divergence*, is given by

$$(\mathrm{d}^* f)(a) = \sum_{b \sim a} f(b, a),$$

where $b \sim a$ denotes that $(a, b) \in E$ or $(b, a) \in E$.

Now to study the cooperative games, given the set of players $[N]$, Stern and Tettenhorst [23] applied the above setup to the *hypercube graph* $G = (V, E)$, where

$$V = 2^{[N]}, \qquad E = \bigl\{ \bigl(S, S \cup \{i\}\bigr) \in V \times V \mid S \subseteq [N] \setminus \{i\},\ i \in [N] \bigr\}.$$

Notice each vertex $S \subseteq [N]$ can correspond to a vertex of the unit hypercube in $\mathbb{R}^N$, and each edge is oriented in the direction of the inclusion $S \hookrightarrow S \cup \{i\}$.

For each $i \in [N]$, let $\mathrm{d}_i \colon \ell^2(V) \to \ell^2(E)$ denote the *partial differential operator*

$$\mathrm{d}_i u\bigl(S, S \cup \{j\}\bigr) = \begin{cases} \mathrm{d}u\bigl(S, S \cup \{i\}\bigr) & \text{if } \ j = i, \\ 0 & \text{if } \ j \neq i. \end{cases}$$

Thus $\mathrm{d}_i v \in \ell^2(E)$ encodes the marginal value contributed by player i to the game v, which is a natural object to consider in view of the Shapley's principle. Indeed, given $v \in \mathcal{G}_N$, Stern and Tettenhorst [23] defined the component game v_i for each

$i \in [N]$ as the unique solution in $\mathcal{G}_N$ to the following *least squares* equation[1]

$$\mathrm{d}^*\mathrm{d}v_i = \mathrm{d}^*\mathrm{d}_i v \tag{1.3}$$

and showed that the component games satisfy some natural properties analogous to the Shapley axioms (see [23, Theorem 3.4]). Moreover, by applying the (pseudo) inverse of the *Laplacian* $\mathrm{d}^*\mathrm{d}$ to (1.3), they provided an explicit formula for v_i (see [23, Theorem 3.11]). In particular, for the *pure bargaining game* $\delta_{[N]}$, defined by

$$\delta_{[N]}([N]) = 1, \quad \delta_{[N]}(S) = 0 \text{ if } S \subsetneq [N]$$

they also showed the complex general formulas in [23, Theorem 3.11] could be simplified (see [23, Theorem 3.13]) due to the substantial symmetry of this game.

Let us, for example, calculate the v_i's for the pure bargaining game $\delta_{[N]}$ when $N = 2, 3$ using the formulas in [23, Theorem 3.13]. For $N = 2$, one can compute

$$v_1(\{1\}) = v_2(\{2\}) = \frac{1}{4},\ v_1(\{2\}) = v_2(\{1\}) = -\frac{1}{4},\ v_1(\{1,2\}) = v_2(\{1,2\}) = \frac{1}{2},$$

and $v_i(\emptyset) = 0$. For $N = 3$, one can compute

$$\begin{gathered}
v_1(\{1\}) = v_2(\{2\}) = v_3(\{3\}) = \frac{1}{12}, \\
v_1(\{2\}) = v_1(\{3\}) = v_2(\{1\}) = v_2(\{3\}) = v_3(\{1\}) = v_3(\{2\}) = -\frac{1}{24}, \\
v_1(\{2,3\}) = v_2(\{1,3\}) = v_3(\{1,2\}) = -\frac{1}{4}, \\
v_1(\{1,2\}) = v_1(\{1,3\}) = v_2(\{1,2\}) = v_2(\{2,3\}) = v_3(\{1,3\}) = v_3(\{2,3\}) = \frac{1}{8}, \\
v_1(\{1,2,3\}) = v_2(\{1,2,3\}) = v_3(\{1,2,3\}) = \frac{1}{3}.
\end{gathered}$$

While we would not try to further compute for $N \geq 4$, we already see, for example, v_i can take negative values even when v is nonnegative, as already noted in [23].

This leads us to ask the following question: What is the economic meaning of the $v_i(S)$ (except for $v_i([N])$, as it coincides with the Shapley value $\phi_i(v)$)?

In this paper we assert that, just as $v_i([N])$, $v_i(S)$ can also represent a fair allocation to the player i of the value $v(S)$ for each coalition S. We shall justify

[1] The equation $\mathrm{d}u = f$ is solvable if only if $f \in \mathcal{R}(\mathrm{d})$. When $f \notin \mathcal{R}(\mathrm{d})$, a least squares solution to $\mathrm{d}u = f$ instead solves $\mathrm{d}u = f_1$ where $f = f_1 + f_2$ with $f_1 \in \mathcal{R}(\mathrm{d})$, $f_2 \in \mathcal{N}(\mathrm{d}^*)$ given by FTLA. By applying d^*, we get $\mathrm{d}^*\mathrm{d}u = \mathrm{d}^*f_1 = \mathrm{d}^*f$. The substitution $u \to v_i$ and $f \to \mathrm{d}_i v$ yields (1.3).

this in two different angles. First, we present a set of new axioms which completely characterizes the component games. Our axioms will be an extension of the Shapley axioms, and may be seen as a completion in view of this characterization of the Hodge-theoretic component games. In particular, the Discussion (iv) in Section 2 explains why the new *reflection axiom* can be regarded natural in view of the Shapley principle, and therefore, why the component value $v_i(S)$ can be interpreted as a fair allocation. Second, we present the value function $V_i(S)$ described by a stochastic path integral which is a natural representation of the total contribution by the player i, on average, toward each coalition S. Hence the value $V_i(S)$ can also be regarded as a fair allocation of $v(S)$ to the player i. Now Theorem 3.1 verifies a remarkable connection between stochastic path integrals and Hodge theory. From the coincidence $V_i = v_i$, we further justify our assertion, namely, $v_i(S) = V_i(S)$ can be viewed a fair allocation for the player i and for each terminal coalition state S.

2. An extension of Shapley axioms

Let $\mathcal{G} = \bigcup_{N \in \mathbb{N}} \mathcal{G}_N$. For $i, j \in [N]$ and $S \subseteq [N]$, define $S^{ij} \subseteq [N]$ by

$$S^{ij} = \begin{cases} S & \text{if } \ S \subseteq [N] \setminus \{i,j\} \ \text{ or } \ \{i,j\} \subseteq S, \\ S \cup \{i\} \setminus \{j\} & \text{if } \ i \notin S \ \text{ and } \ j \in S, \\ S \cup \{j\} \setminus \{i\} & \text{if } \ j \notin S \ \text{ and } \ i \in S. \end{cases}$$

Given $v \in \mathcal{G}_N$, define $v^{ij} \in \mathcal{G}_N$ by $v^{ij}(S) = v(S^{ij})$. Intuitively, the contributions of the players i, j in the game v are interchanged in the game v^{ij}.

Of course, a cooperative game can be considered on any finite set of players M through a bijection $M \hookrightarrow [|M|]$. In this sense, for $v \in \mathcal{G}_N$ and $i \in [N]$, we define v_{-i} to be the restricted game of v on the set of players $[N] \setminus \{i\}$, i.e. $v_{-i}(S) = v(S)$ for all $S \subseteq [N] \setminus \{i\}$. We are ready to describe our extension of Shapley axioms.

Theorem 2.1. *There exists a unique allocation* $v \in \mathcal{G} \mapsto \big(\Phi_i[v]\big)_{i \in \mathbb{N}}$ *satisfying* $\Phi_i[v] \in \mathcal{G}_N$ *with* $\Phi_i[v] \equiv 0$ *if* $i \geq N+1$ *for* $v \in \mathcal{G}_N$*, and moreover the following:*

A1(efficiency): $v = \sum_{i \in \mathbb{N}} \Phi_i[v]$.

A2(symmetry): $\Phi_i[v^{ij}](S^{ij}) = \Phi_j[v](S)$ *for all* $v \in \mathcal{G}_N$*,* $i, j \in [N]$ *and* $S \subseteq [N]$*.*

A3(null-player): *If* $v \in \mathcal{G}_N$ *and* $\mathrm{d}_i v = 0$ *for some* $i \in [N]$*, then* $\Phi_i[v] \equiv 0$*, and*

$$\Phi_j[v](S \cup \{i\}) = \Phi_j[v](S) = \Phi_j[v_{-i}](S) \quad \textit{for all} \ \ j \in [N] \setminus \{i\},\ S \subseteq [N] \setminus \{i\}.$$

A4(linearity): *For any* $v, v' \in \mathcal{G}_N$ *and* $\alpha, \alpha' \in \mathbb{R}$, $\Phi_i[\alpha v + \alpha' v'] = \alpha\Phi_i[v] + \alpha'\Phi_i[v']$.

A5(reflection): *For any* $v \in \mathcal{G}_N$, $i \in [N]$ *and* $S, T \subseteq [N] \setminus \{i\}$, *it holds*

$$\Phi_i[v](S \cup \{i\}) - \Phi_i[v](T \cup \{i\}) = -\big(\Phi_i[v](S) - \Phi_i[v](T)\big).$$

Moreover, the solutions $(v_i)_{i\in[N]}$ *to* (1.3) *satisfy* **A1–A5** *with* $\Phi_i[v] = v_i$.

Discussion. (i) A1, A4 seem natural analogues of corresponding Shapley axioms.
(ii) The condition in A2 is just as if the players i, j exchange their labels. We may interpret as, if the contributions of i, j are interchanged, then so are their payoffs.
(iii) A3 says that if $\mathrm{d}_i v = 0$ then everything is just as if i is not present. In other words, if player i contributes nothing, then the reward of the rest is independent of the participation of the null player i, and hence by efficiency, the player i receives nothing. So $\Phi_i[v] \equiv 0$ is in fact a consequence rather than a part of the axioms.
(iv) Perhaps A5 – which is newly introduced – is the least intuitive of all the axioms. To convince its innocuousness, let $S, T \subseteq [N] \setminus \{i\}$, and consider an arbitrary connected path θ of cooperation process from S to T on the hypercube graph:

$$\theta : X_0 \to X_1 \to \cdots \to X_n$$

where $X_0 = S$, $X_n = T$, and each (X_k, X_{k+1}) is either a forward- or reverse-oriented edge of the hypercube graph. Then there exists its reflection with respect to i:

$$\theta' : X_0' \to X_1' \to \cdots \to X_n'$$

where $X_k' := X_k \cup \{i\}$ if $i \notin X_k$, and $X_k' := X_k \setminus \{i\}$ if $i \in X_k$. Now observe that the contribution of the player i (i.e. the integral of d_i) along the path θ and θ' have the opposite sign, since whenever the player i joins / leaves coalition along θ, i leaves / joins coalition along θ' respectively. In view of the Shapley's principle and the arbitrariness of θ, this is the rationale for us to adopt the reflection axiom. Note that while (1.2) takes into account the coalition processes of joining direction only, our consideration above allows for the coalitions to proceed in either direction.

Now we present a proof of Theorem 2.1.

Proof. We claim that A1–A5 determines the linear operator Φ uniquely (if exists). For each $N \in \mathbb{N}$, define the basis games $\delta_{S,N}$ of $\mathcal{G}_N$ for every $S \subseteq [N]$, $S \neq \emptyset$, by

$$\delta_{S,N}(S) = 1, \quad \delta_{S,N}(T) = 0 \ \text{ if } \ T \neq S.$$

We proceed by an induction on N. The case $N=1$ is already from A1. Suppose the claim holds for $N-1$, so $\Phi_i[\delta_{S,N-1}]$ are determined for all $S \in 2^{[N-1]} \setminus \{\emptyset\}$. Now define the games $\Delta_{(S,S\cup\{i\})}$ in $\mathcal{G}_N$ for each $i \in [N]$, $S \subseteq [N] \setminus \{i\}$, $S \neq \emptyset$, by

$$\Delta_{(S,S\cup\{i\})}(T) = 1 \text{ if } T = S \text{ or } T = S \cup \{i\}, \quad \Delta_{(S,S\cup\{i\})}(T) = 0 \text{ otherwise.}$$

Notice then A3 determines Φ for all $\Delta_{(S,S\cup\{i\})} \in \mathcal{G}_N$. By A4, to prove the claim, it is enough to show A1–A5 can determine Φ for the pure bargaining game $\delta := \delta_{[N],N}$.

By A2, $\sum_{S\subseteq[N]} \Phi_i[\delta](S)$ is constant for all $i \in [N]$, thus it is $1/N$ by A1. Define

$$u_i(S) := \Phi_i[\delta](S) - \frac{1}{N2^N} \text{ for all } S \subseteq [N]$$

so that $u_i(\emptyset) = -\frac{1}{N2^N}$ and $\sum_{S\subseteq[N]} u_i(S) = 0$ for all i. Now observe A5 implies:

$$u_i(S) + u_i(S \cup \{i\}) \text{ is constant for all } S \subseteq [N] \setminus \{i\}, \text{ hence it is zero.}$$

This determines u_i thus $\Phi_i[\delta]$ as follows: suppose $u_i(S)$ has been determined for all i and $|S| \leq k-1$. Let $|T| = k$. Then we have $u_i(T) = -u_i(T \setminus \{i\})$ for all $i \in T$ and it is constant (say c_k) by A2. Then by A1 and A2, $u_j(T) = -\frac{kc_k}{N-k}$ for all $j \notin T$. By induction, the proof of uniqueness of the operator Φ is therefore complete.

It remains to show the solutions $(v_i)_{i\in[N]}$ to (1.3) satisfy **A1–A5** with $\Phi_i[v] = v_i$. Note that $v_i \in \mathcal{G}_N$ is uniquely determined by (1.3) since $\mathcal{N}(\mathrm{d})$ is one-dimensional space spanned by the constant game $\mathbb{1}_N \equiv 1 \in \ell^2(2^{[N]})$, and since $v_i(\emptyset) = 0$.

A4 is clearly satisfied by $(v_i)_i$. To show A1, we compute similarly as in [23]:

$$\mathrm{d}^*\mathrm{d} \sum_{i\in[N]} v_i = \sum_{i\in[N]} \mathrm{d}^*\mathrm{d} v_i = \sum_{i\in[N]} \mathrm{d}^*\mathrm{d}_i v = \mathrm{d}^* \sum_{i\in[N]} \mathrm{d}_i v = \mathrm{d}^*\mathrm{d} v,$$

since $\mathrm{d} = \sum_{i\in[N]} \mathrm{d}_i$. Hence by unique solvability of (1.3), $\sum_{i\in[N]} v_i = v$ as desired.

Next let σ be a permutation of $[N]$. As in [23], let σ act on $\ell^2(V)$ and $\ell^2(E)$ via

$$\sigma v(S) = v(\sigma(S)) \text{ and } \sigma f\big(S, S\cup\{i\}\big) = f\big(\sigma(S), \sigma(S\cup\{i\})\big), \; v \in \ell^2(V), \; f \in \ell^2(E).$$

It is easy to check $\mathrm{d}\sigma = \sigma\mathrm{d}$ and $\mathrm{d}_i\sigma = \sigma\mathrm{d}_{\sigma(i)}$. We also have $\mathrm{d}^*\sigma = \sigma\mathrm{d}^*$, since

$$\langle v, \mathrm{d}^*\sigma f\rangle = \langle \mathrm{d}v, \sigma f\rangle = \langle \sigma^{-1}\mathrm{d}v, f\rangle = \langle \mathrm{d}\sigma^{-1}v, f\rangle = \langle \sigma^{-1}v, \mathrm{d}^*f\rangle = \langle v, \sigma\mathrm{d}^*f\rangle$$

for any $v \in \ell^2(V)$ and $f \in \ell^2(E)$. Now let σ be the transposition of i, j. We have

$$\mathrm{d}^*\mathrm{d}(\sigma v)_i = \mathrm{d}^*\mathrm{d}_i\sigma v = \mathrm{d}^*\sigma\mathrm{d}_j v = \sigma\mathrm{d}^*\mathrm{d}_j v = \sigma\mathrm{d}^*\mathrm{d}v_j = \mathrm{d}^*\mathrm{d}\sigma v_j$$

which shows $(\sigma v)_i = \sigma v_j$ by the unique solvability. Notice this corresponds to A2.

For A3, let $v \in \mathcal{G}_N$, $i \in [N]$, and let $\mathrm{d}_i v = 0$. Then by (1.3) we readily get $v_i \equiv 0$. Fix $j \neq i$, and let $\tilde{\mathrm{d}}$, $\tilde{\mathrm{d}}_j$ be the differential operators restricted on $[N] \setminus \{i\}$, and let $\tilde{v} = v_{-i}$, i.e., $\tilde{v}$ is the restricted game of v on $[N] \setminus \{i\}$. Let $\tilde{v}_j$ be the corresponding component game on $[N] \setminus \{i\}$, solving the defining equation $\tilde{\mathrm{d}}^* \tilde{\mathrm{d}} \tilde{v}_j = \tilde{\mathrm{d}}^* \tilde{\mathrm{d}}_j \tilde{v}$. Finally, in view of A3, define $v_j \in \mathcal{G}_N$ by $v_j = \tilde{v}_j$ on $2^{[N]\setminus\{i\}}$ and $\mathrm{d}_i v_j = 0$. Now observe that A3 will follow if we verify this v_j indeed solves the equation $\mathrm{d}^* \mathrm{d} v_j = \mathrm{d}^* \mathrm{d}_j v$.

To show this, let $S \subseteq [N] \setminus \{i\}$. In fact the following string of equalities holds:

$$\mathrm{d}^* \mathrm{d} v_j(S \cup \{i\}) = \mathrm{d}^* \mathrm{d} v_j(S) = \tilde{\mathrm{d}}^* \tilde{\mathrm{d}} \tilde{v}_j(S) = \tilde{\mathrm{d}}^* \tilde{\mathrm{d}}_j \tilde{v}(S) = \mathrm{d}^* \mathrm{d}_j v(S) = \mathrm{d}^* \mathrm{d}_j v(S \cup \{i\})$$

which simply follows from the definition of the differential operators. For instance

$$\mathrm{d}^* \mathrm{d} v_j(S) = \sum_{T \sim S} \mathrm{d} v_j(T, S) = \sum_{T \sim S,\, T \neq S \cup \{i\}} \mathrm{d} v_j(T, S) = \tilde{\mathrm{d}}^* \tilde{\mathrm{d}} \tilde{v}_j(S)$$

where the second equality is due to $\mathrm{d}_i v_j = 0$. On the other hand, since $j \neq i$,

$$\mathrm{d}^* \mathrm{d}_j v(S) = \sum_{T \sim S} \mathrm{d}_j v(T, S) = \sum_{T \sim S} \mathrm{d}_j \tilde{v}(T, S) = \tilde{\mathrm{d}}^* \tilde{\mathrm{d}}_j \tilde{v}(S).$$

The first and last equalities in the string should now be obvious, verifying A3.

Finally we verify A5. For this, we need to verify the following claim

$$v_i(S) + v_i(S \cup \{i\}) \text{ is constant over all } S \subseteq [N] \setminus \{i\}.$$

Let $S \subseteq [N] \setminus \{i\}$, and recall $\mathrm{d}^* \mathrm{d}_i v(S) = v(S) - v(S \cup \{i\}) = -\mathrm{d}^* \mathrm{d}_i v(S \cup \{i\})$. Hence $\mathrm{d}^* \mathrm{d} v_i(S) + \mathrm{d}^* \mathrm{d} v_i(S \cup \{i\}) = 0$. Define $w_i \in \ell^2(2^{[N]})$ by $w_i(S) = v_i(S \cup \{i\})$ and $w_i(S \cup \{i\}) = v_i(S)$. Then it is clear $\mathrm{d}^* \mathrm{d} v_i(S \cup \{i\}) = \mathrm{d}^* \mathrm{d} w_i(S)$, thus $\mathrm{d}^* \mathrm{d}(v_i + w_i) \equiv 0$, hence $v_i + w_i \in \mathcal{N}(\mathrm{d})$, meaning $v_i + w_i$ is constant. This proves the claim. $\square$

3. A path integral representation of the component games

Recall the Shapley value $\phi_i(v) = v_i([N])$ admits the representation formula (1.2). For $v \in \mathcal{G}_N$, could $v_i(S)$ for $S \subsetneq [N]$ also have an analogous representation?

To give an answer, consider a Markov chain $(X_n)_{n \in \mathbb{N} \cup \{0\}}$ on the state space $2^{[N]}$, $X_0 = \emptyset$, equipped with the transition probability $p_{S,T}$ from S to T as follows:

$$p_{S,T} = 1/N \text{ if } (S, T) \text{ is a forward- or reverse-oriented edge}, \quad p_{S,T} = 0 \text{ otherwise}.$$

Notice the Markov chain describes the canonical coalition progression in which every player can join or leave the current coalition state equally likely at any time.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underlying probability space for formality. For each $S \subseteq [N]$, let $\tau_S = \tau_S(\omega)$ denote the first time the Markov chain $\big(X_n(\omega)\big)_n$ visits S. We define the total contribution of the player i along the sample path $\omega \in \Omega$ toward S by

$$\mathrm{D}_i v(S) = \mathrm{D}_i v(S)(\omega) := \sum_{n=1}^{\tau_S(\omega)} \mathrm{d}_i v\big(X_{n-1}(\omega), X_n(\omega)\big).$$

Finally, we define value functions for each $i \in [N]$ via the following path integral

$$V_i(S) = V_i[v](S) := \int_\Omega \mathrm{D}_i v(S)(\omega) d\mathcal{P}(\omega) = \mathbb{E}[\mathrm{D}_i v(S)].$$

Observe $V_i(S)$ represents the expected total contribution made by player i leading to the state S along all possible coalition paths, and hence can be viewed as a fair allocation of the value $v(S)$ to the player i given that S is the destination state.

Now the following theorem shows V_i is a valid representation of the component game v_i. This gives a further justification for the interpretation of the component game value $v_i(S)$ to be a fair and natural reward allocation for the player i at S.

Theorem 3.1. *For every $v \in \mathcal{G}_N$ and $i \in [N]$ it holds $V_i = v_i$, where $v_i \in \mathcal{G}_N$ is the unique solution to the equation* (1.3) *characterized by A1–A5 in Theorem 2.1.*

Proof. Fix $v \in \mathcal{G}_N$. We may show that $(V_i)_{i\in[N]}$ satisfies A1–A5 with $\Phi_i(v) := V_i$ thanks to Theorem 2.1. A4 may be trivially verified and we omit. For A1, notice

$$\sum_{i\in[N]} \mathrm{D}_i v(S) = \sum_{i\in[N]} \sum_{n=1}^{\tau_S} \mathrm{d}_i v\big(X_{n-1}, X_n\big) = \sum_{n=1}^{\tau_S} \mathrm{d} v\big(X_{n-1}, X_n\big) = v(S)$$

since $\mathrm{d} = \sum_i \mathrm{d}_i$. Hence $\sum_i V_i(S) = \mathbb{E}[\sum_i \mathrm{D}_i v(S)] = v(S)$. This verifies A1.

For $i \neq j$, note that each sample path ω has its counterpart ω^{ij} satisfying

$$X_n(\omega) = S \ \text{ if and only if } \ X_n(\omega^{ij}) = S^{ij} \ \text{ for every } n \in \mathbb{N} \text{ and } S \subseteq [N]$$

and moreover, $\mathrm{D}_i v(S)(\omega) = \mathrm{D}_j v(S^{ij})(\omega^{ij})$. Taking expectation then verifies A2.

For A3, assume $\mathrm{d}_i v = 0$. Then $\mathrm{D}_i v = 0$ readily gives $V_i \equiv 0$. Next we may proceed by an induction on N, so let $(\tilde{X}_n)_n$ denote the Markov chain on the state space $2^{[N]\setminus\{i\}}$ with the analogous transition rates and $(\tilde{V}_i)_i$ be the associated value functions. Observe that $(\tilde{X}_n)_n$ can be embedded in $(X_n)_n$ via the identification

$$\tilde{X}_n := S \in 2^{[N]\setminus\{i\}} \ \text{ if and only if } \ X_n = S \ \text{ or } \ X_n = S \cup \{i\}$$

yielding $\tilde{V}_j(S) = \int_\Omega \sum_{n=1}^{\tau_S \wedge \tau_{S\cup\{i\}}} \mathrm{d}_j v(X_{n-1}, X_n) d\mathcal{P}$ for all $j \in [N]\setminus\{i\}$. Now consider an arbitrary connected finite path $\theta : X_0 \to X_1 \to \cdots \to X_n$ on the hypercube graph where $X_0 = S$ and $X_n = S \cup \{i\}$, or $X_0 = S \cup \{i\}$ and $X_n = S$. Its reversed path is then given by $\theta' : X_0' \to \cdots \to X_n'$ where $X_k' = X_{n-k}$. Finally flip the reversed path with respect to i and get the path $\theta^* : X_0^* \to \cdots \to X_n^*$, where $X_k^* = S \in 2^{[N]\setminus\{i\}}$ if $X_k' = S \cup \{i\}$ and $X_k^* = S \cup \{i\}$ if $X_k' = S$. Observe that the correspondence $\theta \leftrightarrow \theta^*$ is bijective, both starts at S and ends at $S \cup \{i\}$ (or $S \cup \{i\}$ and S), and

$$\sum_{k=1}^{n} \mathrm{d}_j v(X_{k-1}, X_k) = -\sum_{k=1}^{n} \mathrm{d}_j v(X_{k-1}^*, X_k^*)$$

since $\mathrm{d}_i v = 0$. Notice, in view of the path integral formula for V_j, this implies

$$\tilde{V}_j(S) = V_j(S) = V_j(S \cup \{i\}).$$

Induction hypothesis gives $\tilde{V}_j(S) = \tilde{v}_j(S)$ where $\tilde{v}$ is the restriction of v on $2^{[N]\setminus\{i\}}$, and moreover $\tilde{v}_j(S) = v_j(S) = v_j(S \cup \{i\})$ by Theorem 2.1. This verifies A3.

Finally let us verify A5. For any states $S, T \subseteq [N]$, consider the Markov chain $(X_n^S)_n$ whose initial state is $X_0^S = S$ (instead of $\emptyset$). Define analogously $\mathrm{D}_i^S v(T)$ and $V_i^S(T) = \mathbb{E}[\mathrm{D}_i^S v(T)]$ via X^S in place of X. By the correspondence between paths from S to T and their reverse from T to S, it readily follows $V_i^S(T) = -V_i^T(S)$. Now we prove the following transition formula

$$V_i(T) - V_i(S) = V_i^S(T). \tag{3.1}$$

To see this, we compute

$$\begin{aligned} \mathrm{D}_i v(T) - \mathrm{D}_i v(S) &= \sum_{n=1}^{\tau_T} \mathrm{d}_i v\big(X_{n-1}, X_n\big) - \sum_{n=1}^{\tau_S} \mathrm{d}_i v\big(X_{n-1}, X_n\big) \\ &= \mathbf{1}_{\tau_S < \tau_T} \sum_{n=\tau_S+1}^{\tau_T} \mathrm{d}_i v\big(X_{n-1}, X_n\big) - \mathbf{1}_{\tau_T < \tau_S} \sum_{n=\tau_T+1}^{\tau_S} \mathrm{d}_i v\big(X_{n-1}, X_n\big). \end{aligned}$$

By taking expectation, we obtain via the Markov property

$$\mathbb{E}[\mathrm{D}_i v(T)] - \mathbb{E}[\mathrm{D}_i v(S)] = \mathcal{P}(\{\tau_S < \tau_T\}) V_i^S(T) - \mathcal{P}(\{\tau_T < \tau_S\}) V_i^T(S) = V_i^S(T)$$

which proves (3.1). Now for any $S, T \subseteq [N] \setminus \{i\}$, we observe

$$V_i(T \cup \{i\}) - V_i(S \cup \{i\}) = V_i^{S \cup \{i\}}(T \cup \{i\}) = -V_i^S(T) = -\big(V_i(T) - V_i(S)\big)$$

where the middle equality stems from the correspondence of reflected paths with respect to i given in the discussion (iv) in the previous section. This verifies A5. $\square$

A recent work of Lim [14] discusses this theorem in more general graph settings.

References

[1] O. Candogan, I. Menache, A. Ozdaglar, and P. A. Parrilo, *Flows and decompositions of games: harmonic and potential games*, Math. Oper. Res., 36 (2011), pp. 474–503.

[2] J. Castro, D. Gómez, E. Molina, and J. Tejada, *Improving polynomial estimation of the Shapley value by stratified random sampling with optimum allocation*, Comput. Oper. Res., 82 (2017), pp. 180–188.

[3] J. Castro, D. Gómez, and J. Tejada, *Polynomial calculation of the Shapley value based on sampling*, Comput. Oper. Res., 36 (2009), pp. 1726–1730.

[4] X. T. Deng and C. H. Papadimitriou, *On the complexity of cooperative solution concepts*, Math. Oper. Res., 19 (1994), pp. 257–266.

[5] U. Faigle and W. Kern, *The Shapley value for cooperative games under precedence constraints*, Internat. J. Game Theory, 21 (1992), pp. 249–266.

[6] W. V. D. Hodge, *The Theory and Applications of Harmonic Integrals*, Cambridge University Press, England; Macmillan Company, New York, 1941.

[7] X. Jiang, L.-H. Lim, Y. Yao, and Y. Ye, *Statistical ranking and combinatorial Hodge theory*, Math. Program., 127 (2011), pp. 203–244.

[8] E. Kalai and D. Samet, *On weighted Shapley values*, Internat. J. Game Theory, 16 (1987), pp. 205–222.

[9] A. Khmelnitskaya, O. Selçuk, and D. Talman, *The Shapley value for directed graph games*, Oper. Res. Lett., 44 (2016), pp. 143–147.

[10] K. Kodaira, *Harmonic fields in Riemannian manifolds (generalized potential theory)*, Ann. of Math. (2), 50 (1949), pp. 587–665.

[11] G. Koshevoy, T. Suzuki, and D. Talman, *Cooperative games with restricted formation of coalitions*, Discrete Appl. Math., 218 (2017), pp. 1–13.

[12] K. Kultti and H. Salonen, *Minimum norm solutions for cooperative games*, Internat. J. Game Theory, 35 (2007), pp. 591–602.

[13] L.-H. Lim, *Hodge Laplacians on graphs*, SIAM Rev. 62 (2020), no. 3, 685–715.

[14] T. Lim, *Hodge theoretic reward allocation for generalized cooperative games on graphs*, arXiv preprint.

[15] H. Peters, *Game theory. A multi-leveled approach.* Springer-Verlag, 2008.

[16] A.E. Roth, *The Shapley Value as a von Neumann-Morgenstern Utility*, Econometrica, 45 (1977), pp. 657–664.

[17] A.E. Roth, *A.E Roth (Ed.), The Shapley Value: Essays in Honor of Lloyd S. Shapley*, Cambridge Univ. Press, New York (1988).

[18] L. M. Ruiz, F. Valenciano, and J. M. Zarzuelo, *The family of least square values for transferable utility games*, Games Econ. Behav., (1998) 109–130.

[19] L. S. Shapley, *Additive and non-additive set functions*, ProQuest LLC, Ann Arbor, MI, 1953. Thesis (Ph.D.)–Princeton University.

[20] L. S. Shapley, *A value for n-person games*, in Contributions to the theory of games, vol. 2, Annals of Mathematics Studies, no. 28, 1953, pp. 307–317.

[21] L. S. Shapley, *Stochastic games*, Proc. Nat. Acad. Sci. 39 (1953), 1095–1100.

[22] L. S. Shapley, *Utility comparison and the theory of games* (reprint), Bargaining and the theory of cooperative games: John Nash and beyond, 235–247, Edward Elgar, Cheltenham, 2010.

[23] A. Stern and A. Tettenhorst, *Hodge decomposition and the Shapley value of a cooperative game*, Games and Economic Behavior,113 (2019) 186–198.