STOCHASTIC INTEGRAL REPRESENTATION OF SOLUTIONS TO HODGE THEORETIC POISSON'S EQUATIONS ON GRAPHS, AND COOPERATIVE VALUE ALLOCATION OF SHAPLEY AND NASH

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ABSTRACT. The fundamental connection between stochastic differential equations (SDEs) and partial differential equations (PDEs) has found numerous applications in diverse fields. We explore a similar link between stochastic calculus and combinatorial PDEs on graphs with Hodge structure, by showing that the solution to the Hodge-theoretic Poisson's equation on graphs allows for a stochastic integral representation driven by a canonical time-reversible Markov chain. When the underlying graph has a hypercube structure, we further show that the solution to the Poisson's equation can be fully characterized by five properties, which can be thought of as a completion of the Lloyd Shapley's four axioms.

Keywords: Hodge decomposition, Poisson's equation, least squares, weighted graph, stochastic integral representation, Markov chain, time-reversibility, cooperative game, Shapley value, Shapley formula, Nash solution, Kohlberg and Neyman's value
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1. INTRODUCTION AND OUR CONTRIBUTION

Consider the graph $G = (\Xi, E)$, where Ξ is a finite set of vertices and E is the set of (forward- or positively- oriented) edges. To begin, we define the weighted inner product space of functions $\ell^2(\Xi)$, $\ell^2(E)$. Let ρ and λ be strictly positive weight functions on Ξ and E, respectively, and set $\lambda(T,S) = \lambda(S,T)$ by convention for any $(S,T) \in E$. Note that at most one of (S,T) and (T,S) is in E for $S,T \in \Xi$.

Denote by $\ell^2_\rho(\Xi)$ the space of functions $\Xi \to \mathbb{R}$ equipped with the inner product

$$\langle u, v\rangle_\rho := \sum_{S\in\Xi} \rho(S)u(S)v(S). \tag{1.1}$$

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Denote by $\ell^2_\lambda(E)$ the space of functions $E \to \mathbb{R}$ equipped with the inner product

$$\langle f, g\rangle_\lambda \coloneqq \sum_{(S,T)\in E} \lambda(S,T) f(S,T) g(S,T) \tag{1.2}$$

with the Hodge structure $f(T,S) \coloneqq -f(S,T)$ for the reverse-oriented edge (T,S). Thus every $f \in \ell^2_\lambda(E)$ is defined not only on all the "positively oriented" edges, i.e., the edges in E, but also on their "negatively oriented" reverse. For $S, T \in \Xi$, we declare there exists a (forward- or reverse-) edge (S,T) if and only if $\lambda(S,T) > 0$. Then we say that the weighted graph G is *connected* if for any $S, T \in \Xi$ there exists a chain of (forward- or reverse-) edges $\big((S_k, S_{k+1})\big)_{k=0}^{n-1}$ with $S_0 = S$, $S_n = T$.

Next, we endow G with a Hodge differential structure. For $v \in \ell^2_\rho(\Xi)$, define a linear operator $\mathrm{d}\colon \ell^2_\rho(\Xi) \to \ell^2_\lambda(E)$, the *gradient*, by

$$\mathrm{d}v(S,T) \coloneqq v(T) - v(S). \tag{1.3}$$

The adjoint d^*, (negative) *divergence*, satisfies for all $v \in \ell^2_\rho(\Xi)$ and $f \in \ell^2_\lambda(E)$

$$\langle \mathrm{d}v, f\rangle_\lambda = \langle v, \mathrm{d}^* f\rangle_\rho. \tag{1.4}$$

To find the explicit formula for d^*, let $(\mathbb{1}_S)_{S\in\Xi}$ be the standard basis of $\ell^2(\Xi)$, where $\mathbb{1}_S(T) = 1$ if $T = S$ and otherwise 0. Then it is easy to see that

$$\mathrm{d}^* f(S) = \frac{\langle \mathbb{1}_S, \mathrm{d}^* f\rangle_\rho}{\rho(S)} = \frac{\langle \mathrm{d}\mathbb{1}_S, f\rangle_\lambda}{\rho(S)} = \sum_{T\sim S} \frac{\lambda(T,S)}{\rho(S)} f(T,S) \tag{1.5}$$

where $T \sim S$ denotes $\lambda(S,T) > 0$, i.e., S and T are adjacent. The *graph Laplacian* is now defined by the operator $\mathrm{d}^*\mathrm{d}$. In this context, combinatorial Hodge decomposition simply corresponds to the Fundamental Theorem of Linear Algebra:

$$\ell^2_\rho(\Xi) = \mathcal{R}(\mathrm{d}^*) \oplus \mathcal{N}(\mathrm{d}), \qquad \ell^2_\lambda(E) = \mathcal{R}(\mathrm{d}) \oplus \mathcal{N}(\mathrm{d}^*), \tag{1.6}$$

where $\mathcal{R}(\cdot)$, $\mathcal{N}(\cdot)$ stand for the range and nullspace respectively.

The primary focus of this paper is the combinatorial Poisson's equation:

$$\mathrm{d}^*\mathrm{d}v = \mathrm{d}^* f. \tag{1.7}$$

Our first main result is the stochastic integral representation of the solution v. Let us first note that, while the weight ρ affects the divergence d^* as seen in (1.5), it has no effect on the solution to (1.7). We also observe uniqueness of the solution.

Lemma 1.1. *i) The solutions v to* (1.7) *do not depend on the choice of ρ.*
ii) If the graph G is connected, any two solutions u, v to (1.7) *differ by a constant.*

On the other hand, the solution does depend on λ. In light of this lemma, we will simply assume $\rho \equiv 1$ from now on. Now we introduce the stochastic integral representation for the solution to (1.7). Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given an edge weight λ, consider the canonical Markov chain $(X_n^U)_{n\in\mathbb{N}_0}$ on the state space Ξ with $X_0 = U$, equipped with the transition probability $p_{S,T}$ from a state S to T defined by

$$p_{S,T} = \frac{\lambda(S,T)}{\sum_{U\sim S}\lambda(S,U)} \ \text{ if } \ T \sim S, \quad p_{S,T} = 0 \ \text{ if } \ T \nsim S. \tag{1.8}$$

The Markov chain (1.8) is known to be *time-reversible*, which means that there exists a stationary distribution $\pi = (\pi_S)_{S\in\Xi}$, satisfying

$$\pi_S p_{S,T} = \pi_T p_{T,S} \ \text{ for all } \ S, T \in \Xi. \tag{1.9}$$

One important implication is that every loop and its inverse have the same probability of being realized, that is (see, e.g., Ross [32])

$$p_{S,S_1}p_{S_1,S_2}\cdots p_{S_{n-1},S_n}p_{S_n,S} = p_{S,S_n}p_{S_n,S_{n-1}}\cdots p_{S_2,S_1}p_{S_1,S}. \tag{1.10}$$

Let $(\Omega, \mathcal{F}, \mathcal{P})$ denote the probability space for the Markov chain. For each $S, T \in \Xi$ and $\omega \in \Omega$, let $\tau_{S,T} = \tau_{S,T}(\omega) \in \mathbb{N}_0$ denote the first (random) time the Markov chain $\big(X_n^S(\omega)\big)_n$ visits T. Given a "marginal contribution measure" $f \in \ell^2(E)$, we define the total contribution along the sample path ω traveling from S to T by

$$\mathcal{I}_f^S(T) = \mathcal{I}_f^S(T)(\omega) := \sum_{n=1}^{\tau_{S,T}(\omega)} f\big(X_{n-1}^S(\omega), X_n^S(\omega)\big). \tag{1.11}$$

We can imagine that the space Ξ represents all possible *project progress states*, and $f(U,V)$ represents the contribution value of a player/employee when the state moves from U to a neighbor state V. Given that the project state has progressed from S to T along the path ω, (1.11) represents the player's total contribution throughout the progression. In this case, the edge weight function λ determines which direction the project is likely to take via (1.8).

Now we define the value function via the following stochastic path integral:

$$V_f^S(T) := \int_\Omega \mathcal{I}_f^S(T)(\omega)d\mathcal{P}(\omega) = \mathbb{E}[\mathcal{I}_f^S(T)]. \tag{1.12}$$

$V_f^S(T)$ represents a player's expected total contribution if the state advances from S to T, where f represents the player's marginal contribution for each transition.

Our first main result is the following.

Theorem 1.2. *Let $f \in \ell^2_\lambda(E)$ and let the Markov chain* (1.8) *be defined on a weighted graph (G, λ). Then V_f^S solves the Poisson's equation*

$$\mathrm{d}^*\mathrm{d}V_f^S = \mathrm{d}^* f \tag{1.13}$$

on the connected component of G that contains the initial state S.

The fundamental connection between stochastic differential equations (SDEs) and partial differential equations (PDEs) established by Kolmogorov, Feynman, Kac, and many others has found numerous applications in diverse fields. The author hopes that Theorem 1.2 will spark a similar link between stochastic calculus and combinatorial PDEs on graphs with Hodge structure. In establishing this result, the author notices with interest that the time-reversibility property of Markov processes on graphs appears to play an analogous role to the martingale property on Euclidean spaces. As we will see, the proof of Theorem 1.2 will be purely combinatorial, with the time-reversibility (1.10) playing an important role.

We also note that Theorem 1.2 shows that when calculating the value function V_f for a player whose marginal contribution is given by f, one can instead solve the Poisson's equation (1.7), which can be easily done using least squares solvers. The solution to (1.7), on the other hand, can be approximated by simulating the Markov chain (1.8) and calculating the contribution aggregator (1.12).

The following example in financial decision making will demonstrate an interesting application and relevance of the framework we have introduced thus far.

Example 1.3 (Entrepreneur's revenue problem)**.** *Let Ξ represent the project state space in which the manager wishes to achieve the project completion state $F \in \Xi$. Let $v : \Xi \to \mathbb{R}$ denote the manager's revenue, i.e., $v(U)$ represents the manager's revenue if the project reaches the state U. Let $[N] = \{1, ..., N\}$ denote the employees with their marginal contribution measures $f_1, ..., f_N \in \ell^2(E)$. Because it is her contribution and share, the manager must pay $f_i(S,T)$ to the employee i at each state transition from S to T. Thus, the manager's surplus in this single transition is $v(T) - v(S) - \sum_i f_i(S,T)$. Now the manager's revenue problem is: What is the* *manager's expected revenue if they begin at the initial project state (say O, where $v(O) = 0$) and the manager's goal is to reach the project completion state F?*

Observe the answer is $v(F) - \sum_i V_{f_i}^O(F)$, where $V_{f_i}^O$ is defined by the stochastic integral given the marginal contribution measure f_i as in (1.12), *for each $i \in [N]$. (So if this is negative, the manager may decide not to begin the project at all.)*

Furthermore, in the middle of the project, the manager may want to recalculate her expected gain or loss. That is, suppose the current project status is T, and they arrived at T via a specific path ω, and thus the manager has paid the payoffs — the path integrals (1.11)*— to the employees. The manager may now wish to calculate the expected gain if she decides to proceed from T to F. This is now provided by*

$$v(F) - v(T) - \sum_i V_{f_i}^T(F),$$

and the manager can make decisions based on the expected revenue data. Finally, Theorem 1.2 allows us to calculate $V_{f_i}^T$ in terms of the system of equations (1.13).

Next, in order to describe our second main result, we will focus our discussion on the *hypercube graph*, or *coalition game graph*, with uniform weights $\rho \equiv 1$, $\lambda \equiv 1$:

$$\Xi = 2^{[N]}, \quad E = \big\{ \big(S, S \cup \{i\}\big) \in \Xi \times \Xi \mid S \subseteq [N] \setminus \{i\},\ i \in [N] \big\}, \tag{1.14}$$

where $[N] = \{1, 2, ..., N\}$ represents the *players* of the *coalition games*

$$\mathcal{G}_N = \{v : 2^{[N]} \to \mathbb{R} \mid v(\emptyset) = 0\}.$$

Note that a coalition game v is simply a (value) function on the subsets of $[N]$, where each $S \subseteq [N]$ represents a coalition of players in S, and $v(S)$ represents the value assigned to the coalition S, with the null coalition $\emptyset$ receiving zero value. Notice each coalition $S \subseteq [N]$ can correspond to a vertex of the unit hypercube in $\mathbb{R}^N$, and each edge is oriented in the direction of the inclusion $S \hookrightarrow S \cup \{i\}$.

For each $i \in [N]$, let $\mathrm{d}_i \colon \ell^2(\Xi) \to \ell^2(E)$ denote the *partial differential operator*

$$\mathrm{d}_i v\big(S, S \cup \{j\}\big) = \begin{cases} \mathrm{d}v\big(S, S \cup \{i\}\big) & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases} \tag{1.15}$$

$\mathrm{d}_i v \in \ell^2(E)$ encodes the marginal value contributed by player i to the game v. Given $v \in \mathcal{G}_N$, Stern and Tettenhorst [39] defined the *component game* v_i for each

$i \in [N]$ as the unique solution in $\mathcal{G}_N$ to the following form of Poisson's equation

$$\mathrm{d}^*\mathrm{d}v_i = \mathrm{d}^*\mathrm{d}_i v. \tag{1.16}$$

According to our first main result, Theorem 1.2, the component game value $v_i(S)$ for each coalition $S \subseteq [N]$ can be interpreted as the player i's expected total contribution — thus her *fair share* — if the state advances from $\emptyset$ to S, where the player i's marginal contribution for each transition is defined by $\mathrm{d}_i v$. Now inspired by the well-known *Shapley axioms*, which characterize the *Shapley value* $v_i([N])$ for every $i \in [N]$ (see Section 1.1 for details), our second main result will provide an axiomatic characterization of the values $v_i(S)$ for every $i \in [N]$ and $S \subseteq [N]$. That is, we will look for conditions that will completely determine the solutions to (1.16). For this, let $\mathcal{G} = \bigcup_{N \in \mathbb{N}} \mathcal{G}_N$. For $i, j \in [N]$ and $S \subseteq [N]$, define $S^{ij} \subseteq [N]$ by

$$S^{ij} = \begin{cases} S & \text{if } \ S \subseteq [N] \setminus \{i,j\} \ \text{ or } \ \{i,j\} \subseteq S, \\ S \cup \{i\} \setminus \{j\} & \text{if } \ i \notin S \ \text{ and } \ j \in S, \\ S \cup \{j\} \setminus \{i\} & \text{if } \ j \notin S \ \text{ and } \ i \in S. \end{cases}$$

Given $v \in \mathcal{G}_N$ and $i, j \in [N]$, we define $v^{ij} \in \mathcal{G}_N$ by $v^{ij}(S) = v(S^{ij})$. Intuitively, the contributions of the players i, j in the game v are interchanged in the game v^{ij}.

Of course, a coalition game can be considered on any finite set of players M through a bijection $M \hookrightarrow [|M|]$. In this sense, we define v_{-i} to be the restricted game of v on the set of players $[N] \setminus \{i\}$, i.e., $v_{-i}(S) = v(S)$ for all $S \subseteq [N] \setminus \{i\}$. We are now prepared to describe our second main result.

Theorem 1.4. *There exists a unique allocation map $v \in \mathcal{G} \mapsto \big(\Phi_i[v]\big)_{i \in \mathbb{N}}$ satisfying $\Phi_i[v] \in \mathcal{G}_N$ with $\Phi_i[v] \equiv 0$ for $i > N$ if $v \in \mathcal{G}_N$, and also the following conditions:*

A1(efficiency): $v = \sum_{i \in \mathbb{N}} \Phi_i[v]$.

A2(symmetry): $\Phi_i[v^{ij}](S^{ij}) = \Phi_j[v](S)$ *for all* $v \in \mathcal{G}_N$, $i, j \in [N]$ *and* $S \subseteq [N]$.

A3(null-player): *If* $v \in \mathcal{G}_N$ *and* $\mathrm{d}_i v = 0$ *for some* $i \in [N]$, *then* $\Phi_i[v] \equiv 0$, *and*

$$\Phi_j[v](S \cup \{i\}) = \Phi_j[v](S) = \Phi_j[v_{-i}](S) \quad \text{for all} \ \ j \in [N] \setminus \{i\},\ S \subseteq [N] \setminus \{i\}.$$

A4(linearity): *For any* $v, v' \in \mathcal{G}_N$ *and* $\alpha, \alpha' \in \mathbb{R}$, $\Phi_i[\alpha v + \alpha' v'] = \alpha \Phi_i[v] + \alpha' \Phi_i[v']$.

A5(reflection): *For any* $v \in \mathcal{G}_N$ *and* $S \subseteq [N] \setminus \{i, j\}$ *with* $i \neq j$, *it holds*

$$\Phi_i[v](S \cup \{i,j\}) - \Phi_i[v](S \cup \{i\}) = -\big(\Phi_i[v](S \cup \{j\}) - \Phi_i[v](S)\big).$$

Furthermore, the solution $v_i \in \mathcal{G}_N$ *to* (1.16) *satisfy* **A1–A5** *with the identification* $\Phi_i[v] = v_i$. *In other words,* **A1–A5** *characterizes the solutions* $\{v_i\}_{i\in[N]}$ *to* (1.16).

In fact, this computer-scientific result is inspired by Lloyd Shapley's value allocation theory for cooperative games, which we will discuss in the following section. In light of this characterization of the solutions to the Hodge-theoretic Poisson's equation (1.16), our conditions A1–A5 can be viewed as a completion of Shapley's original four axioms. Furthermore, in Section 3, we will demonstrate how our value allocation operator $V_i := V^{\emptyset}_{\mathrm{d}_i v}$ defined in (1.12) can generalize Nash's and Kohlberg–Neyman's value allocation scheme for *strategic cooperative games.*

1.1. Shapley's four axioms of value allocation for coalition games.

Shapley considered the question of how to split the grand coalition value $v([N])$ among the players for a given game $v \in \mathcal{G}_N$. It is determined uniquely by the following result.

Theorem 1.5 (Shapley [34])**.** *There exists a unique allocation* $v \in \mathcal{G}_N \mapsto \big(\phi_i(v)\big)_{i\in[N]}$ *satisfying the following conditions:*

- efficiency: $\sum_{i\in[N]} \phi_i(v) = v([N])$.
- symmetry: $v\big(S \cup \{i\}\big) = v\big(S \cup \{j\}\big)$ *for all* $S \subseteq [N] \setminus \{i, j\}$ *yields* $\phi_i(v) = \phi_j(v)$.
- null-player: $v\big(S \cup \{i\}\big) - v(S) = 0$ *for all* $S \subseteq [N] \setminus \{i\}$ *yields* $\phi_i(v) = 0$.
- linearity: $\phi_i(\alpha v + \alpha' v') = \alpha\phi_i(v) + \alpha'\phi_i(v')$ *for all* $\alpha, \alpha' \in \mathbb{R}$ *and* $v, v' \in \mathcal{G}_N$.

Moreover, this allocation is given by the following explicit formula:

$$\phi_i(v) = \sum_{S\subseteq[N]\setminus\{i\}} \frac{|S|!\big(N-1-|S|\big)!}{N!}\Big(v\big(S\cup\{i\}\big) - v(S)\Big). \tag{1.17}$$

The four conditions listed above are commonly referred to as the *Shapley axioms*. According to [39], [efficiency] means that the value obtained by the grand coalition is fully distributed among the players, [symmetry] means that equivalent players receive equal amounts, [null-player] means that a player who contributes no marginal value to any coalition receives nothing, and [linearity] means that the allocation is linear in game values. And (1.17) is referred to as the *Shapley formula.*

(1.17) can be rewritten also according to [39]: Suppose the players form the grand coalition by joining, one-at-a-time, in the order defined by a permutation σ of $[N]$. That is, player i joins immediately after the coalition $S^\sigma_i = \big\{j \in [N] : \sigma(j) < \sigma(i)\big\}$ has formed, contributing marginal value $v\big(S^\sigma_i \cup \{i\}\big) - v(S^\sigma_i)$. Then $\phi_i(v)$ is the average marginal value contributed by player i over all $N!$ permutations σ, i.e.,

$$\phi_i(v) = \frac{1}{N!}\sum_{\sigma}\Big(v\big(S_i^\sigma \cup \{i\}\big) - v(S_i^\sigma)\Big) = \frac{1}{N!}\sum_{\sigma} \mathrm{d}_i v(S_i^\sigma). \tag{1.18}$$

The well-known glove game below explains the formula (1.18) in a simple context. Let $N = 3$, and suppose that player 1 has a left-hand glove, while players 2 and 3 each have a right-hand glove. A pair of gloves has value 1, while unpaired gloves have no value. In other words, $v(S) = 1$ if $S \subseteq N$ contains player 1 and at least one of players 2 or 3, and $v(S) = 0$ otherwise. The Shapley values are given by:

$$\phi_1(v) = \frac{2}{3}, \qquad \phi_2(v) = \phi_3(v) = \frac{1}{6}.$$

This is easily seen from (1.18): player 1 contributes marginal value 0 when joining the coalition first (2 of 6 permutations) and marginal value 1 otherwise (4 of 6 permutations), so $\phi_1(v) = \frac{2}{3}$. Efficiency and symmetry yield $\phi_2(v) = \phi_3(v) = \frac{1}{6}$.

Recently, [39] showed that the component game $v_i \in \mathcal{G}_N$ solving (1.16) satisfies

$$v_i([N]) = \phi_i(v) \ \text{ for every } \ i \in [N], \tag{1.19}$$

thereby obtaining a new characterization of the Shapley value as the value of the grand coalition in each player's component game.

However, each v_i is not just defined at the state $[N]$ but at any state $S \subseteq [N]$. This leads us to ask: *What is the economic significance of $v_i(S)$, when $S \subsetneq [N]$?*

The explicit calculations that follow make this question a little more interesting. Let $\delta_{[N]} \in \mathcal{G}_N$ be the *pure bargaining game*, defined by $\delta_{[N]}([N]) = 1$ and $\delta_{[N]}(S) = 0$ if $S \subsetneq [N]$. One can calculate the component games $(v_i)_i$ for the pure bargaining game $\delta_{[N]}$ using the formulas in [39, Theorem 3.13]. For $N = 2$, one can compute

$$v_1(\{1\}) = v_2(\{2\}) = \frac{1}{4}, \ \ v_1(\{2\}) = v_2(\{1\}) = -\frac{1}{4}, \ \ v_1(\{1,2\}) = v_2(\{1,2\}) = \frac{1}{2}.$$

For $N = 3$, $v_i(\{1,2,3\}) = \frac{1}{3}$ for all $i \in \{1,2,3\}$ is clearly the Shapley value, and

$$v_1(\{1\}) = v_2(\{2\}) = v_3(\{3\}) = \frac{1}{12}, \ \ v_1(\{2,3\}) = v_2(\{1,3\}) = v_3(\{1,2\}) = -\frac{1}{4},$$
$$v_1(\{2\}) = v_1(\{3\}) = v_2(\{1\}) = v_2(\{3\}) = v_3(\{1\}) = v_3(\{2\}) = -\frac{1}{24},$$
$$v_1(\{1,2\}) = v_1(\{1,3\}) = v_2(\{1,2\}) = v_2(\{2,3\}) = v_3(\{1,3\}) = v_3(\{2,3\}) = \frac{1}{8}.$$

Even for such a simple game $\delta_{[N]}$, the formulas in [39, Theorem 3.13] become increasingly complicated as N increases, and we hardly find any pattern in the values. However, we can observe that v_i can take negative values even if v is nonnegative.

Theorem 1.2 is a result of our effort to find an answer to the preceding question. Notice that, by Theorem 1.2 and (1.19), the Shapley value (1.18) and the value of our allocation operator at $[N]$ coincide, that is (see (1.12)):

$$\phi_i(v) = V_i([N]), \text{ where } V_i(S) := \mathbb{E}\left[\sum_{n=1}^{\tau_{\emptyset,S}} \mathrm{d}_i v\big(X_{n-1}^{\emptyset}, X_n^{\emptyset}\big)\right] \text{ for } S \subseteq [N]. \tag{1.20}$$

The summation formulas in (1.18) and (1.20), on the other hand, appear quite different. While (1.18) consists of a finite sum along $N!$ paths in increasing order driven by permutations σ, the paths ω in (1.20) are driven by the Markov chain

$$p_{S,T} = 1/N \text{ if } (S,T) \text{ is a forward or reverse edge}, \quad p_{S,T} = 0 \text{ otherwise}, \tag{1.21}$$

describing the canonical coalition progression in which every player has an equal chance of joining or leaving the current coalition state at any time. The sum in (1.20) (or, more generally, (1.12)) is, in particular, infinite. While the Shapley formula (1.18) cannot easily be extended to other partial coalitions $S \subsetneq [N]$, our value function (1.20) immediately extends to all states and provides its significance as a fair allocation of the collaborative reward $v(S)$ when $S \subsetneq [N]$. In this sense, (1.20) can be thought of as completing the Shapley formula.

However, we continue to ask the following: If the Shapley axioms can characterize the Shapley value $V_i([N])$, are there conditions that can characterize the values $V_i(S)$ for all states S? In other words, are there conditions that can characterize the solutions to the Poisson equation $\mathrm{d}^*\mathrm{d}v_i = \mathrm{d}^*\mathrm{d}_i v$ for any $v \in \mathcal{G}_N$? And, if they do exist, will they have corresponding economic interpretation as the Shapley axioms?

Theorem 1.4, our second main result, now provides an answer by extending the Shapley axioms through A1–A5. Let us go over these in more detail below.

Discussion for A1–A5. A1 and A4 are natural analogues of the corresponding Shapley axioms. The condition in A2 is the same as if the players i, j switched labels. We can interpret this as follows: if the contributions of i, j are interchanged, so are their payoffs. A3 states that if $\mathrm{d}_i v = 0$, everything is the same as if i is not present. In other words, if player i contributes nothing, the reward of the rest is independent of the null player i's participation, and thus the player i receives nothing by efficiency. So $\Phi_i[v] \equiv 0$ is a result rather than a part of the axioms.

We've seen that A1–A4 are a natural extension of the Shapley axioms to deal with different numbers of players N and coalitions S, as well as their symmetric counterpart S^{ij}. In particular, A1–A4 will determine the Shapley value $V_i([N])$. However, A1–A4 appear to be insufficient to fully determine $V_i(S)$ for all coalitions $S \subseteq [N]$, and our observation is that the reflection condition A5 appears to be the key to complement A1–A4, on which we will now elaborate. In A5, by fixing i and repeatedly adding players j in S, we find that A5 is equivalent to:[1]

A5'(reflection): For any $v \in \mathcal{G}_N$, $i \in [N]$ and $S, T \subseteq [N] \setminus \{i\}$, it holds

$$\Phi_i[v](S \cup \{i\}) - \Phi_i[v](T \cup \{i\}) = -\big(\Phi_i[v](S) - \Phi_i[v](T)\big). \tag{1.22}$$

A5' is indeed inspired by the stochastic integral representation of the value function V_i (1.20), which we will now discuss. Let $S, T \subseteq [N] \setminus \{i\}$, and consider an arbitrary connected path ω of coalition process from S to T on the hypercube graph (1.14):

$$\omega : X_0 \to X_1 \to \cdots \to X_n$$

where $X_0 = S$, $X_n = T$, and each (X_k, X_{k+1}) is either a forward- or reverse-oriented edge of the hypercube graph. Then the *reflection of* ω *with respect to* i is given by

$$\omega' : X'_0 \to X'_1 \to \cdots \to X'_n$$

where $X'_k := X_k \cup \{i\}$ if $i \notin X_k$, and $X'_k := X_k \setminus \{i\}$ if $i \in X_k$. We observe that the total contribution of the player i (that is, the sum of d_i's) along the paths ω and ω' has the opposite sign, because whenever the player i joins or leaves coalition along ω, i leaves or joins coalition along ω'. By intergrating over all ω connecting S and T, we finally arrive at (1.22). We emphasize the distinction once more: whereas the Shapley formula (1.18) considers coalition processes in the joining direction only, our path integral allows coalitions to proceed in either direction, which eventually yields a complete characterization of the values $V_i(S)$ for all coalitions S thanks to A5. In this sense, A1–A5 can be thought of as completing the Shapley axioms.

[1]The author thanks Ari Stern for pointing out this equivalence.

The remainder of the paper is structured as follows. In Section 2, we account for the background. In Section 3, we delve deeper into the economic significance of our findings. More specifically, we briefly introduce Nash's and Kohlberg–Neyman's value allocation scheme for the *strategic cooperative games*, and explain how their axiomatic notion of value can be reinterpreted and extended to all states $S \subseteq [N]$ in terms of our value allocation operator. Section 4 will present proofs of the results.

2. Historical account and relevant work

In the 1940s, Richard Feynman discovered that the Schrödinger equation, the differential equation governing the wave function of a quantum-mechanical system, could be solved by averaging over paths, which led to a far-reaching reformulation of quantum theory in terms of his path integrals [12]. Then came the Feynman–Kac formula, which rigorously proves the real case of Feynman's path integrals and was inspired when Mark Kac attended a Feynman's seminar in 1947. Meanwhile, Andrei Kolmogorov published a foundational paper [20] on continuous time Markov processes in 1931, and his equations were later dubbed the Kolmogorov forward and backward equations by William Feller [11]. Since then, much progress has been made in the fundamental connection between SDEs and PDEs, and we refer to Stroock and Varadhan [40], Karatzas and Shreve [15], and Lawler [22] for a comprehensive account. We note that the martingale property of stochastic processes, and thus the vanishing drift terms in the Itô calculus, appears to play a central role in theories related to the Feynman–Kac formula and more general connections between SDEs and PDEs. On the other hand, the martingale property is nowhere used in the derivation of Theorem (1.2), but time-reversibility appears to be critical. We only dealt with the Poisson's equation in Theorem (1.2), but we believe that there should be a much richer connection between various combinatorial PDEs and stochastic path-integrals on graphs with Hodge structure, which is the author's main research agenda. For example, the author would like to know if there is a PDE whose solution represents the variance of the path-integral (1.11).

Mathematical finance is one of the major and rapidly growing fields where the SDE–PDE connection is especially important, with the Black–Scholes–Merton option pricing formula [6, 26, 27] serving as a representative example. In this regard, we refer to Shreve [37, 38], Karatzas and Shreve [16], and the references therein.

Extensive research on the cooperative and noncooperative games has been inspired by and evolved from the pioneering study by Shapley [33–36], and his original four axioms have been followed by many further considerations and variants, e.g., Young [41] and Chun [3]. The A3 condition in Theorem 1.4 may be seen as an extension of the "null player out" property in Derks and Haller [10], by noting that A3 now applies for all coalitions S. The use of graph structures to describe cooperation, as well as their orders and constraints, is credited to Myerson [28, 29].

Our contribution is that while previous axiomatic approaches could only characterize the Shapley value $V_i([N])$, our axioms, especially when complemented by A5, can now characterize the value allocation $V_i(S)$ for all $S \subseteq [N]$. That is, our A1–A5 can now characterize the solution to the Poisson's equation of the form (1.16). Another contribution is that we now introduce completely general graphs to represent general cooperation processes (e.g., Example 1.3), whereas previous works mostly focused on coalition graphs that only describe the order of the coalition, i.e., the order in which the players join the coalition. Previous research in cooperative game theory has primarily focused on how to create sound "axioms" that can characterize value allocation. Now, our introduction of general game graphs implies that, rather than a set of axioms, the main task for a given strategic and cooperative situation may be to set up a suitable game graph with the likelihood of the progression direction. If this task is completed successfully, our value allocation operator (1.12) will calculate a natural and fair value allocation for each participant. This idea is highlighted in Section 3, which reinterprets and extends Nash's and Kohlberg–Neyman's value allocation for strategic cooperative games.

Recently, the combinatorial Hodge decomposition has been applied to game theory in a variety of contexts, e.g., noncooperative games (Candogan et al. [2]), cooperative games (Stern and Tettenhorst [39]), and ranking of social preferences (Jiang et al. [14]). Lim [23] provides an introduction to Hodge theory on graphs, while Hodge [13] and Kodaira [17] provide general and profound Hodge theory.

Another important recent development is the *mean field game theory*, the study of strategic decision making by interacting agents in large populations; see, e.g., Cardaliaguet et al. [7], Acciaio et al. [1], Bayraktar et al. [4, 5], Possamaï et al. [31], Lacker and Soret [21]. Carmona and Delarue [8, 9] provides a detailed account.

3. Dynamic interpretation and extension of Nash's and Kohlberg–Neyman's value allocation for strategic games

According to Kohlberg and Neyman [18], a *strategic game* is a model for a multiperson competitive interaction. Each player chooses a strategy, and the combined choices of all the players determine a payoff to each of them. A problem of interest in game theory is the following: How to evaluate, in advance of playing a game, the economic worth of a player's position? A "value" is a general solution, that is, a method for evaluating the worth of any player in a given strategic game.

In this section, we briefly introduce Nash's and Kohlberg–Neyman's value, and explain how their axiomatic notion of value can be represented by our value allocation operator (1.20), and as a result, can readily be generalized to all coalitions.

According to [18], a strategic game is defined by a triple $G = ([N], A, g)$, where

- $[N] = \{1, 2, ..., N\}$ is a finite set of players,
- A^i is the finite set of player i's pure strategies, and $A = \prod_{i=1}^N A^i$,
- $g^i : A \to \mathbb{R}$ is player i's payoff function, and $g = (g^i)_{i\in[N]}$.

The same notation, g, is used to denote the linear extension

- $g^i : \Delta(A) \to \mathbb{R}$,

where for any set K, $\Delta(K)$ denotes the probability distributions on K.

For each coalition $S \subseteq [N]$, we also denote

- $A^S = \prod_{i\in S} A^i$, and
- $X^S = \Delta(A^S)$ (correlated strategies of the players in S).

Let $\mathbb{G}([N])$ be the set of all N-player strategic games. Consider $\gamma : \mathbb{G}([N]) \to \mathbb{R}^N$ that associates with any strategic game an allocation of payoffs to the players. Now, Kohlberg and Neyman [18] proposed a set of axioms for characterizing γ, the core concept of which is the following definition of the *threat power* of coalition S:

$$(\delta G)(S) := \max_{x\in X^S} \min_{y\in X^{[N]\setminus S}} \left(\sum_{i\in S} g^i(x, y) - \sum_{i\notin S} g^i(x, y) \right). \tag{3.1}$$

The threat power of S (to the other party $[N] \setminus S$) can be read as the maximum difference between the sum of the players' payoffs in S and the sum of the other party's payoffs, regardless of what collective strategies the other party employs.

Then Kohlberg and Neyman demonstrated that the axioms of *Efficiency* (the sum of all players' payoffs, i.e., $(\delta G)([N])$, is fully distributed among the players), *Balanced threats* (see below), *Symmetry* (equivalent players receive equal amounts), *Null player* (a player having no strategic impact on players' payoffs has zero value), and *Additivity* (the allocation is additive on strategic games) uniquely determine an allocation γ; see [18] for details. Moreover, such allocation γ is a generalization of the Nash solution for two-person games [30] into N-person games, according to [18]. Now among the axioms, the axiom of balanced threats asserts the following:

- If $(\delta G)(S) = 0$ for all $S \subseteq [N]$, then $\gamma_i = 0$ for all $i \in [N]$.

Namely, if no coalition S has threat power over the other party, the allocation is zero for all players. From now on let $\gamma = (\gamma_1, ..., \gamma_N)$ denote the unique allocation determined by the above five axioms. [18] also provided an explicit formula for γ:

$$\gamma_i G = \frac{1}{N!} \sum_\sigma (\delta G)\big(\bar{S}_i^\sigma\big), \tag{3.2}$$

where the summation is over all permutations σ of the set $[N]$, S_i^σ is the subset consisting of those $j \in [N]$ that precede i in the ordering σ, and $\bar{S}_i^\sigma := S_i^\sigma \cup \{i\}$. Now we focus on (3.2) and manipulate it as follows. By minimax principle, it is easily seen that $(\delta G)(S) = -(\delta G)([N] \setminus S)$. This *antisymmetry* implies

$$\begin{aligned}
\gamma_i G &= \frac{1}{N!} \sum_\sigma \frac{(\delta G)\big(\bar{S}_i^\sigma\big) - (\delta G)\big([N] \setminus \bar{S}_i^\sigma\big)}{2} \\
&= \frac{1}{2N!} \sum_\sigma (\delta G)\big(\bar{S}_i^\sigma\big) - \frac{1}{2N!} \sum_\sigma (\delta G)\big([N] \setminus \bar{S}_i^\sigma\big) \\
&= \frac{1}{2N!} \sum_\sigma (\delta G)\big(\bar{S}_i^\sigma\big) - \frac{1}{2N!} \sum_\sigma (\delta G)\big(S_i^\sigma\big). \qquad (3.3)
\end{aligned}$$

Motivated by this, let us define the coalition game $v = v_G : 2^{[N]} \to \mathbb{R}$ as follows:

$$v(S) := \frac{(\delta G)(S) + (\delta G)([N])}{2} = \frac{(\delta G)([N]) - (\delta G)([N] \setminus S)}{2}. \tag{3.4}$$

The value function $v(S)$ may be interpreted as the grand coalition value $(\delta G)([N])$ subtracted by the threat power of the other party $[N] \setminus S$, with a factor of $1/2$.

By the fact that the value function v is a translation of $\delta G/2$, we have

$$\mathrm{d}_i v(S_i^\sigma) = v(\bar{S}_i^\sigma) - v(S_i^\sigma) = \frac{(\delta G)\big(\bar{S}_i^\sigma\big) - (\delta G)\big(S_i^\sigma\big)}{2}.$$

In view of (3.3), we arrive at the following alternative expression for $\gamma_i G$:

$$\gamma_i G = \frac{1}{N!} \sum_{\sigma} \mathrm{d}_i v(S_i^\sigma).$$

We observe that this is the Shapley value (1.18) for the coalition game $v = v_G$. Then we recall that [39] defined the component game v_i for each $i \in [N]$ as the unique solution in $\mathcal{G}_N$ to the equation $\mathrm{d}^*\mathrm{d}v_i = \mathrm{d}^*\mathrm{d}_i v$, and showed that the component game value at the grand coalition coincides with the Shapley value, that is, $v_i([N]) = \gamma_i G$ in this context. With this, Theorem 1.2 now allows us to conclude the following.

Theorem 3.1 (Stochastic integral extension of Nash's and Kohlberg–Neyman's value)**.** *Given a strategic game $G \in \mathbb{G}([N])$, let $v \in \mathcal{G}(2^{[N]})$ be the coalition game defined as in* (3.4)*. Let the hypercube graph* (1.14) *be equipped with constant weight $\lambda \equiv 1$, and let $(X_n)_{n\in\mathbb{N}_0}$ be the canonical Markov chain* (1.21) *with $X_0 = \emptyset$. Then for each player $i \in [N]$ and every coalition $S \subseteq [N]$, the value allocation operator*

$$V_i(S) = \mathbb{E}\left[\sum_{n=1}^{\tau_{\emptyset,S}} \mathrm{d}_i v\big(X_{n-1}, X_n\big)\right]$$

extends Nash's and Kohlberg–Neyman's value in the sense that $V_i([N]) = \gamma_i G$. Furthermore, the conditions A1–A5 in Theorem (1.4) *characterizes the value $V_i(S)$ for all coalitions $S \subseteq [N]$, including the Kohlberg and Neyman's value $\gamma_i G$.*

Proof. Stern and Tettenhorst [39] showed $v_i([N]) = \gamma_i G$, where $v_i \in \mathcal{G}_N$ is the solution to (1.16). Theorem 1.2 and Lemma 1.1(ii) then yields $v_i = V_i$ on $2^{[N]}$. □

We note that Kohlberg and Neyman also introduce the concept of *Bayesian games*, which is a game of incomplete information in the sense that the players do not know the true payoff functions, but only receive a signal that is correlated with the payoff functions; see [18] for details. However, the threat power, $(\delta_B G)(S)$, of a coalition S in the Bayesian game G remains antisymmetric, i.e., $(\delta_B G)(S) = -(\delta_B G)([N] \setminus S)$, and the value allocation also satisfies the representation formula (3.2). As a result, we can conclude that the value of Bayesian games still admits the stochastic path-integral extension for all coalitions, as shown in Theorem 3.1. We refer to [18, 19] for a nice review of the historical development of the ideas around the notion of value, as well as several applications to various economic models.

4. Proofs

We present proofs of the results. Results are restated for the reader's convenience.

Lemma 4.1. *i) The solutions v to* (1.7) *do not depend on the choice of ρ.*
ii) If the graph G is connected, any two solutions u, v to (1.7) *differ by a constant.*

Proof. i) $(\mathrm{d}^*\mathrm{d}v - \mathrm{d}^* f)(S) = \frac{1}{\rho(S)} \sum_{T\sim S} \lambda(T,S)[v(S) - v(T) - f(T,S)]$ shows that $(\mathrm{d}^*\mathrm{d}v - \mathrm{d}^* f)(S) = 0$ if and only if $\sum_{T\sim S} \lambda(T,S)[v(S) - v(T) - f(T,S)] = 0$, showing there is no dependence on ρ. ii) Observe that the connectedness of G implies that the nullspace $\mathcal{N}(\mathrm{d})$ is one-dimensional, spanned by the constant game $\mathbb{1}$, defined by $\mathbb{1}(S) := 1$ for all $S \in \Xi$. Now if $\mathrm{d}^*\mathrm{d}u = \mathrm{d}^*\mathrm{d}v$, then we have $u - v \in \mathcal{N}(\mathrm{d})$. □

Now we'll look at the proof of Theorem 1.2. This necessitates the development of a transition formula for the value function. The fact that the Markov chain is irreducible and thus visits every state infinitely many times is used implicitly, and furthermore, the time-reversibility of the Markov chain seems crucial to the proofs.

Lemma 4.2. *Let (G, λ) be any connected weighted graph. For any $S, T, U \in \Xi$ and $f \in \ell^2_\lambda(E)$, we have $V_f^U(T) - V_f^U(S) = V_f^S(T)$.*

Proof. We'll first prove a special case $V_f^S(T) = -V_f^T(S)$. Consider a general finite sample path ω of the Markov chain (1.8) starting at S, visiting T, then returning to S (this happens with probability 1). We can split this journey into four subpaths:
ω_1: the path returns to S $m \in \mathbb{N}_0$ times without visiting T,
ω_2: the path begins at S and ends at T without returning to S,
ω_3: the path returns to T $n \in \mathbb{N}_0$ times without visiting S,
ω_4: the path begins at T and ends at S without returning to T.
Thus $\omega = \omega_1 \circ \omega_2 \circ \omega_3 \circ \omega_4$ is the concatenation of the ω_i's, and the probability $\mathcal{P}(\omega)$ of this finite sample path being realized satisfies $\mathcal{P}(\omega) = \mathcal{P}(\omega_1)\mathcal{P}(\omega_2)\mathcal{P}(\omega_3)\mathcal{P}(\omega_4)$.

Now consider a pairing ω' of ω as follows: let ω_1^{-1} be the reversed path of ω_1, that is, if ω_1 visits $T_0 \to T_1 \to \cdots \to T_k$ (where $T_0 = T_k = S$ for ω_1), then ω_1^{-1} visits $T_k \to \cdots \to T_0$. Recall $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1^{-1})$ due to the time-reversibility (1.10). Now define $\omega' := \omega_1^{-1} \circ \omega_2 \circ \omega_3^{-1} \circ \omega_4$. This is another general sample path starting at S, visiting T, then returning to S. Then we have $\mathcal{P}(\omega) = \mathcal{P}(\omega')$, and moreover,

$$\mathcal{I}_f^S(T)(\omega) + \mathcal{I}_f^S(T)(\omega') = 2 \sum_{n=1}^{\tau_{S,T}(\omega_2)} f\big(X_{n-1}^S(\omega_2), X_n^S(\omega_2)\big),$$

because the loops ω_1 and ω_1^{-1} aggregate f with opposite signs, hence they cancel out in the above sum. Now consider $\tilde{\omega} := \omega_3 \circ \omega_2^{-1} \circ \omega_1 \circ \omega_4^{-1}$ and $\tilde{\omega}' := \omega_3^{-1} \circ \omega_2^{-1} \circ \omega_1^{-1} \circ \omega_4^{-1}$. $(\tilde{\omega}, \tilde{\omega}')$ then represents a pair of general sample paths starting at T, visiting S, then returning to T. We then calculate

$$\begin{aligned}\mathcal{I}_f^T(S)(\tilde{\omega}) + \mathcal{I}_f^T(S)(\tilde{\omega}') &= 2 \sum_{n=1}^{\tau_{T,S}(\omega_2^{-1})} f\big(X_{n-1}^T(\omega_2^{-1}), X_n^T(\omega_2^{-1})\big) \\ &= -2 \sum_{n=1}^{\tau_{S,T}(\omega_2)} f\big(X_{n-1}^S(\omega_2), X_n^S(\omega_2)\big) \\ &= -(\mathcal{I}_f^S(T)(\omega) + \mathcal{I}_f^S(T)(\omega'))\end{aligned}$$

because $f(U, V) = -f(V, U)$ for any edge (U, V). Due to the generality of the pair (ω, ω') and its counterpart $(\tilde{\omega}, \tilde{\omega}')$, and $\mathcal{P}(\omega) = \mathcal{P}(\omega') = \mathcal{P}(\tilde{\omega}) = \mathcal{P}(\tilde{\omega}')$ from the reversibility (1.10), the desired identity $V_f^S(T) = -V_f^T(S)$ now follows by integration. Next, in order to show $V_f^U(T) - V_f^U(S) = V_f^S(T)$, we proceed

$$\begin{aligned}\mathcal{I}_f^U(T) - \mathcal{I}_f^U(S) &= \sum_{n=1}^{\tau_{U,T}} f\big(X_{n-1}^U, X_n^U\big) - \sum_{n=1}^{\tau_{U,S}} f\big(X_{n-1}^U, X_n^U\big) \\ &= \mathbf{1}_{\tau_{U,S} < \tau_{U,T}} \sum_{n=\tau_{U,S}+1}^{\tau_{U,T}} f\big(X_{n-1}^U, X_n^U\big) - \mathbf{1}_{\tau_{U,T} < \tau_{U,S}} \sum_{n=\tau_{U,T}+1}^{\tau_{U,S}} f\big(X_{n-1}^U, X_n^U\big).\end{aligned}$$

By taking expectation, we obtain the following via the Markov property

$$\begin{aligned}\mathbb{E}[\mathcal{I}_f^U(T)] - \mathbb{E}[\mathcal{I}_f^U(S)] &= \mathcal{P}(\{\tau_{U,S} < \tau_{U,T}\}) V_f^S(T) - \mathcal{P}(\{\tau_{U,T} < \tau_{U,S}\}) V_f^T(S) \\ &= V_f^S(T)\end{aligned}$$

which proves the transition formula $V_f^U(T) - V_f^U(S) = V_f^S(T)$. □

Theorem 4.3. *Let $f \in \ell_\lambda^2(E)$ and let the Markov chain* (1.8) *be defined on a weighted graph (G, λ). Then V_f^S solves the Poisson's equation*

$$\mathrm{d}^* \mathrm{d} V_f^S = \mathrm{d}^* f \tag{4.1}$$

on the connected component of G that contains the initial state S.

Proof. By Lemma 1.1, we can set $\rho \equiv 1$. Let $\{T_1, ..., T_n\}$ be the set of all vertices adjacent to T (i.e., either (T, T_k) or (T_k, T) is in E), and set $\Lambda_T = \sum_{k=1}^n \lambda(T, T_k)$.

Assume $S, T, T_1, ..., T_n$ lie in a connected component of G. By (1.5), (1.8), we have

$$\mathrm{d}^*f(T)/\Lambda_T = \sum_{k=1}^{n} p_{T,T_k} f(T_k, T), \text{ and} \tag{4.2}$$

$$\mathrm{d}^*\mathrm{d}V_f^S(T)/\Lambda_T = \sum_{k=1}^{n} p_{T,T_k}\big(V_f^S(T) - V_f^S(T_k)\big) = \sum_{k=1}^{n} p_{T,T_k} V_f^{T_k}(T) \tag{4.3}$$

where the last equality is from Lemma 4.2. Now observe that we can interpret (4.3) as the aggregation (1.12) of path integrals of f (1.11) along all loops beginning and ending at T, but in this aggregation of f we do not take into account the first move from T to T_k, since this first move is described by the transition rate p_{T,T_k} and not driven by $V_f^{T_k}$. Meanwhile, if we aggregate path integrals of f for all loops emanating from T, we get zero due to the reversibility (1.10). Hence we conclude:

$$\begin{aligned}
0 &= \text{aggregation of path integrals of } f \text{ along all loops emanating from } T\\
&= \text{aggregation of path integrals of } f \text{ along all loops except the first moves}\\
&\quad + \text{aggregation of path integrals of } f \text{ for all first moves from } T\\
&= \sum_{k=1}^{n} p_{T,T_k} V_f^{T_k}(T) + \sum_{k=1}^{n} p_{T,T_k} f(T, T_k)\\
&= \mathrm{d}^*\mathrm{d}V_f^S(T)/\Lambda_T - \mathrm{d}^*f(T)/\Lambda_T,
\end{aligned}$$

yielding $\mathrm{d}^*\mathrm{d}V_f^S(T) = \mathrm{d}^*f(T)$. This completes the proof. □

Theorem 4.4. *There exists a unique allocation map $v \in \mathcal{G} \mapsto \big(\Phi_i[v]\big)_{i\in\mathbb{N}}$ satisfying $\Phi_i[v] \in \mathcal{G}_N$ with $\Phi_i[v] \equiv 0$ for $i > N$ if $v \in \mathcal{G}_N$, and also the following conditions:*
A1(efficiency): $v = \sum_{i\in\mathbb{N}} \Phi_i[v]$.
A2(symmetry): $\Phi_i[v^{ij}](S^{ij}) = \Phi_j[v](S)$ *for all* $v \in \mathcal{G}_N$, $i, j \in [N]$ *and* $S \subseteq [N]$.
A3(null-player): *If* $v \in \mathcal{G}_N$ *and* $\mathrm{d}_i v = 0$ *for some* $i \in [N]$, *then* $\Phi_i[v] \equiv 0$, *and*

$$\Phi_j[v](S \cup \{i\}) = \Phi_j[v](S) = \Phi_j[v_{-i}](S) \ \ \textit{for all} \ \ j \in [N] \setminus \{i\},\, S \subseteq [N] \setminus \{i\}.$$

A4(linearity): *For any* $v, v' \in \mathcal{G}_N$ *and* $\alpha, \alpha' \in \mathbb{R}$, $\Phi_i[\alpha v + \alpha' v'] = \alpha\Phi_i[v] + \alpha'\Phi_i[v']$.
A5(reflection): *For any* $v \in \mathcal{G}_N$ *and* $S \subseteq [N] \setminus \{i, j\}$ *with* $i \neq j$, *it holds*

$$\Phi_i[v](S \cup \{i, j\}) - \Phi_i[v](S \cup \{i\}) = \Phi_i[v](S) - \Phi_i[v](S \cup \{j\}).$$

Furthermore, the solution $v_i \in \mathcal{G}_N$ *to* (1.16) *satisfy* **A1–A5** *with the identification* $\Phi_i[v] = v_i$. *In other words,* **A1–A5** *characterizes the solutions* $\{v_i\}_{i\in[N]}$ *to* (1.16).

Proof. Recall that A5 is equivalent to A5', i.e., for any $S, T \subseteq [N] \setminus \{i\}$, it holds

$$\Phi_i[v](S \cup \{i\}) - \Phi_i[v](T \cup \{i\}) = -\big(\Phi_i[v](S) - \Phi_i[v](T)\big). \tag{4.4}$$

We claim that A1–A5' determines the linear operator Φ uniquely (if exists). For each $N \in \mathbb{N}$, define the basis games $\delta_{S,N}$ of $\mathcal{G}_N$ for every $S \subseteq [N]$, $S \neq \emptyset$, by

$$\delta_{S,N}(S) = 1, \quad \delta_{S,N}(T) = 0 \text{ if } T \neq S.$$

We proceed by an induction on N. The case $N = 1$ is already from A1. Suppose the claim holds for $N-1$, so $\Phi_i[\delta_{S,N-1}]$ are determined for all $S \in 2^{[N-1]} \setminus \{\emptyset\}$. Now define the games $\Delta_{(S,S\cup\{i\})} \in \mathcal{G}_N$ for each $i \in [N]$, $S \subseteq [N] \setminus \{i\}$, $S \neq \emptyset$, by

$$\Delta_{(S,S\cup\{i\})}(T) = 1 \text{ if } T = S \text{ or } T = S \cup \{i\}, \quad \Delta_{(S,S\cup\{i\})}(T) = 0 \text{ otherwise.}$$

Notice then A3 determines Φ for all $\Delta_{(S,S\cup\{i\})} \in \mathcal{G}_N$. By A4, to prove the claim, it is enough to show A1–A5' can determine Φ for the pure bargaining game $\delta := \delta_{[N],N}$.

By A2, $\sum_{S\subseteq[N]} \Phi_i[\delta](S)$ is constant for all $i \in [N]$, thus it is $1/N$ by A1. Define

$$u_i(S) := \Phi_i[\delta](S) - \frac{1}{N2^N} \text{ for all } S \subseteq [N]$$

so that $u_i(\emptyset) = -\frac{1}{N2^N}$ and $\sum_{S\subseteq[N]} u_i(S) = 0$ for all i. Now observe A5' implies:

$$u_i(S) + u_i(S \cup \{i\}) \text{ is constant for all } S \subseteq [N] \setminus \{i\}, \text{ hence it is zero.}$$

This determines u_i thus $\Phi_i[\delta]$ as follows: suppose $u_i(S)$ has been determined for all i and $|S| \leq k-1$. Let $|T| = k$. Then we have $u_i(T) = -u_i(T\setminus\{i\})$ for all $i \in T$ and it is constant (say c_k) by A2. Then by A1 and A2, $u_j(T) = -\frac{kc_k}{N-k}$ for all $j \notin T$. By induction, the proof of uniqueness of the operator Φ is therefore complete.

It remains to show the solutions $(v_i)_{i\in[N]}$ to (1.16) satisfy A1–A5' with $\Phi_i[v] = v_i$. A4 is clearly satisfied by $(v_i)_i$. To show that A1 is satisfied, we compute

$$\mathrm{d}^*\mathrm{d} \sum_{i\in[N]} v_i = \sum_{i\in[N]} \mathrm{d}^*\mathrm{d}v_i = \sum_{i\in[N]} \mathrm{d}^*\mathrm{d}_i v = \mathrm{d}^* \sum_{i\in[N]} \mathrm{d}_i v = \mathrm{d}^*\mathrm{d}v,$$

since $\mathrm{d} = \sum_{i\in[N]} \mathrm{d}_i$. Hence by unique solvability of (1.16), $\sum_{i\in[N]} v_i = v$ as desired. Next let σ be a permutation of $[N]$. As in [39], let σ act on $\ell^2(2^{[N]})$ and $\ell^2(E)$ via

$$\sigma v(S) = v(\sigma(S)) \text{ and } \sigma f\big(S, S\cup\{i\}\big) = f\big(\sigma(S), \sigma(S\cup\{i\})\big), \ v \in \ell^2(2^{[N]}), \ f \in \ell^2(E).$$

It is easy to check $\mathrm{d}\sigma = \sigma\mathrm{d}$ and $\mathrm{d}_i\sigma = \sigma\mathrm{d}_{\sigma(i)}$. We also have $\mathrm{d}^*\sigma = \sigma\mathrm{d}^*$, since

$$\langle v, \mathrm{d}^*\sigma f\rangle = \langle \mathrm{d}v, \sigma f\rangle = \langle \sigma^{-1}\mathrm{d}v, f\rangle = \langle \mathrm{d}\sigma^{-1}v, f\rangle = \langle \sigma^{-1}v, \mathrm{d}^* f\rangle = \langle v, \sigma\mathrm{d}^* f\rangle$$

for any $v \in \ell^2(2^{[N]})$, $f \in \ell^2(E)$. Now let σ be the transposition of i, j. We have

$$\mathrm{d}^*\mathrm{d}(\sigma v)_i = \mathrm{d}^*\mathrm{d}_i\sigma v = \mathrm{d}^*\sigma\mathrm{d}_j v = \sigma\mathrm{d}^*\mathrm{d}_j v = \sigma\mathrm{d}^*\mathrm{d}v_j = \mathrm{d}^*\mathrm{d}\sigma v_j$$

which shows $(\sigma v)_i = \sigma v_j$ by the unique solvability. Notice this corresponds to A2.

For A3, let $v \in \mathcal{G}_N$, $i \in [N]$, and assume $\mathrm{d}_i v = 0$. Then from (1.16) we readily get $v_i \equiv 0$. Fix $j \neq i$, and let $\tilde{\mathrm{d}}$, $\tilde{\mathrm{d}}_j$ be the differential operators restricted on $2^{[N]\setminus\{i\}}$, and set $\tilde{v} = v_{-i}$, i.e., $\tilde{v}$ is the restriction of v on $2^{[N]\setminus\{i\}}$. Let $\tilde{v}_j$ be the corresponding component game on $2^{[N]\setminus\{i\}}$, solving the defining equation $\tilde{\mathrm{d}}^*\tilde{\mathrm{d}}\tilde{v}_j = \tilde{\mathrm{d}}^*\tilde{\mathrm{d}}_j\tilde{v}$. Finally, in view of A3, define $v_j \in \mathcal{G}_N$ by $v_j = \tilde{v}_j$ on $2^{[N]\setminus\{i\}}$ and $\mathrm{d}_i v_j = 0$. Now observe that A3 will follow if we can verify that this v_j indeed solves the equation $\mathrm{d}^*\mathrm{d}v_j = \mathrm{d}^*\mathrm{d}_j v$.

To show this, let $S \subseteq [N] \setminus \{i\}$. In fact the following string of equalities holds:

$$\mathrm{d}^*\mathrm{d}v_j(S \cup \{i\}) = \mathrm{d}^*\mathrm{d}v_j(S) = \tilde{\mathrm{d}}^*\tilde{\mathrm{d}}\tilde{v}_j(S) = \tilde{\mathrm{d}}^*\tilde{\mathrm{d}}_j\tilde{v}(S) = \mathrm{d}^*\mathrm{d}_j v(S) = \mathrm{d}^*\mathrm{d}_j v(S \cup \{i\})$$

which simply follows from the definition of the differential operators. For instance

$$\mathrm{d}^*\mathrm{d}v_j(S) = \sum_{T\sim S} \mathrm{d}v_j(T, S) = \sum_{T\sim S,\, T\neq S\cup\{i\}} \mathrm{d}v_j(T, S) = \tilde{\mathrm{d}}^*\tilde{\mathrm{d}}\tilde{v}_j(S)$$

where the second equality is due to $\mathrm{d}_i v_j = 0$. On the other hand, since $j \neq i$,

$$\mathrm{d}^*\mathrm{d}_j v(S) = \sum_{T\sim S} \mathrm{d}_j v(T, S) = \sum_{T\sim S} \tilde{\mathrm{d}}_j\tilde{v}(T, S) = \tilde{\mathrm{d}}^*\tilde{\mathrm{d}}_j\tilde{v}(S).$$

The first and last equalities in the string should now be obvious, verifying A3.

Finally we verify A5'. For this, we need to verify the following claim

$$v_i(S) + v_i(S \cup \{i\}) \text{ is constant over all } S \subseteq [N] \setminus \{i\}.$$

Let $S \subseteq [N]\setminus\{i\}$, and recall $\mathrm{d}^*\mathrm{d}_i v(S) = v(S) - v(S\cup\{i\}) = -\mathrm{d}^*\mathrm{d}_i v(S\cup\{i\})$. Hence $\mathrm{d}^*\mathrm{d}v_i(S) + \mathrm{d}^*\mathrm{d}v_i(S \cup \{i\}) = 0$. Define $w_i \in \ell^2(2^{[N]})$ by $w_i(S) = v_i(S \cup \{i\})$ and $w_i(S \cup \{i\}) = v_i(S)$ for all $S \subseteq [N] \setminus \{i\}$. Then clearly $\mathrm{d}^*\mathrm{d}v_i(S \cup \{i\}) = \mathrm{d}^*\mathrm{d}w_i(S)$ and $\mathrm{d}^*\mathrm{d}v_i(S) = \mathrm{d}^*\mathrm{d}w_i(S \cup \{i\})$. Thus $\mathrm{d}^*\mathrm{d}(v_i + w_i) \equiv 0$, hence $v_i + w_i \in \mathcal{N}(\mathrm{d})$, meaning that $v_i + w_i$ is constant. This proves the claim, hence the theorem. $\square$

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