

OPTIMAL EXERCISE DECISION OF AMERICAN OPTIONS UNDER MODEL UNCERTAINTY

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ABSTRACT. Given the marginal distribution information of the underlying asset price at two future times T_1 and T_2 , we consider the problem of determining a model-free upper bound on the price of a class of American options that must be exercised at either T_1 or T_2 . The model uncertainty consistent with the given marginal information is described as the martingale optimal transport problem. We show that any option exercise scheme associated with any market model that jointly maximizes the expected option payoff must be nonrandomized if the American option payoff satisfies a suitable convexity condition and the model-free price upper bound and its relaxed version coincide. The latter condition is desired to be removed under appropriate conditions on the cost and marginals.

Keywords: Robust finance, American option, Hedging, Martingale, Optimal transport, Duality, Dual attainment, Infinite-dimensional linear programming

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1. INTRODUCTION

This paper was mainly inspired by Hobson and Norgilas [18], Aksamit, Deng, Obłój and Tan [1], as well as Beiglböck and Juillet [8] and Beiglböck, Nutz and Touzi [9]. A related problem in continuous time setup was studied in Bayraktar, Cox and Stoev [2]. We consider two future times $0 < T_1 < T_2$ and an asset price process (X, Y) , where X, Y represents the asset price at time T_1, T_2 , respectively. Let $\mathcal{P}(\mathcal{X})$ denote the set of all probability measures/distributions over a set $\mathcal{X}$ with finite first moment. Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ be probability measures in convex order:

$$\mu \preceq_c \nu \text{ if } \mu(f) \leq \nu(f) \text{ for every convex function } f \text{ on } \mathbb{R},$$

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Version 1 of the paper posted on arXiv had an incorrect Proposition 2.1, which was used to erroneously derive the equation $P_c = \bar{P}_c$. The proposition was removed in Ver 2, and the main theorem now assumes the equation. We would like to find sufficient conditions for the equation.

where $\mu(f) := \mathbb{E}_\mu[f(X)] = \int f(x)\mu(dx)$. We consider market models that are defined by the following set of martingale transports from μ to ν :

$$\mathcal{M}(\mu, \nu) = \{\pi \in \mathcal{P}(\mathbb{R}^2) \mid \pi = \text{Law}(X, Y), \mathbb{E}_\pi[Y|X] = X, \text{Law}(X) = \mu, \text{Law}(Y) = \nu\}.$$

In finance, each $\pi \in \mathcal{M}(\mu, \nu)$ represents a feasible joint law of the price (X, Y) given the marginal information μ, ν in the (two-period) market, under which (X, Y) is a martingale, written as $\mathbb{E}_\pi[Y|X] = X$. It is well known that the condition $\mu \preceq_c \nu$ is equivalent to $\mathcal{M}(\mu, \nu) \neq \emptyset$. We refer to [10, 11, 13, 14] for further background.

We consider the cost function which describes an American option payoff

$$(1.1) \quad c = (c_1, c_2) = (c_1(x), c_2(x, y)), \quad c_1, c_2 \in \mathbb{R},$$

such that if an obligee (option holder) selects c_1 , she receives the payout $c_1(X)$, otherwise she receives the payout $c_2(X, Y)$. Thus, in the former case, her payout is determined at time 1, whereas it is determined at time 2 in the latter. We assume she can make this choice conditional on the price $X = x$, and that she can also randomize (or split) her choice, represented by a Borel function $s : \mathbb{R} \rightarrow [0, 1]$. This means that given $X = x$, she exercises c_1 with probability (or proportion) $s(x)$, otherwise c_2 with probability $1 - s(x)$. Given a function $s : \mathbb{R} \rightarrow \mathbb{R}$ and a measure μ on $\mathbb{R}$, let the measure $s\mu$ be given by $s\mu(B) = \int_B s(x)\mu(dx)$. Since μ is fixed, the choice of a randomization s is equivalent to the choice of $0 \leq \mu_1 \leq \mu$,¹ such that with $\mu_2 := \mu - \mu_1$, $s_1 := s$, $s_2 := 1 - s$ equals the Radon–Nikodym derivative $\frac{d\mu_1}{d\mu}, \frac{d\mu_2}{d\mu}$ μ -a.s., respectively. This leads us to consider the optimization problem

$$(1.2) \quad P_c := \sup_{\pi \in \mathcal{M}(\mu, \nu)} \sup_{\mu_1 \leq \mu} \mathbb{E}_{\gamma_1}[c_1] + \mathbb{E}_{\gamma_2}[c_2],$$

where for a given $\pi = \pi_x \otimes \mu \in \mathcal{M}(\mu, \nu)$,² we define $\gamma_l = \pi_x \otimes \mu_l$, $l = 1, 2$, such that $\gamma_1 + \gamma_2 = \pi$ and that γ_1 and γ_2 share the same kernel $\{\pi_x\}_x$ inherited from π .

In view of the obligor (the person responsible for the payment of the option), a solution (π, μ_1) to (1.2) represents a worst-possible market scenario π combined with the option exercise scheme μ_1 , yielding the maximum expected payout P_c .

We will assume the following regularity condition on c throughout the paper.

¹All measures/distributions in this paper are assumed to be non-negative.

²Any $\pi = \text{Law}(X, Y) \in \mathcal{P}(\mathbb{R}^2)$, representing the joint law of the random variables X and Y , can be written as $\pi = \pi_x \otimes \text{Law}(Y|X)$, where $\pi_x \in \mathcal{P}(\mathbb{R})$ is called a kernel of π with respect to $\text{Law}(X)$. π_x represents the conditional distribution of Y given $X = x$, i.e., $\pi_x(B) = \mathcal{P}(Y \in B | X = x)$ for all Borel set $B \subseteq \mathbb{R}$. Note that $\pi = \pi_x \otimes \mu \in \mathcal{M}(\mu, \nu)$ iff $\int y \pi_x(dy) = x$ μ -a.e. x .

[A] Throughout the paper, we assume that c_1, c_2 are continuous, $\mu \preceq_c \nu$, and that the marginals μ, ν satisfy the following condition: there exist continuous functions $v \in L^1(\mu)$, $w \in L^1(\nu)$ such that $|c_1| + |c_2| \leq v(x) + w(y)$. Note that this implies

$$\left| \sum_l \mathbb{E}_{\gamma_l}[c_l] \right| \leq \sum_l \mathbb{E}_{\gamma_l}[|c_l|] \leq \sum_l \mathbb{E}_{\pi}[|c_l|] \leq \mu(v) + \nu(w) < \infty \text{ for any } \pi \in \mathcal{M}(\mu, \nu).$$

This in turn implies that the problem (1.2) is attained (i.e., admits an optimizer) by a standard argument in the calculus of variations [22].

[18] considered a specific cost called an American put, whose payoff is given by

$$(1.3) \quad c_1(x) = (K_1 - x)^+, \quad c_2(x, y) = c_2(y) = (K_2 - y)^+, \quad K_1 > K_2,$$

and considered those option exercise schemes which are *pure*, or *non-randomized*; that is, [18] assumed that the obligee can only choose a Borel set $B \subseteq \mathbb{R}$ in which she selects c_1 if $x \in B$ and c_2 otherwise. In terms of μ_1 , notice that this is equivalent to the statement that μ_1 and μ_2 are mutually disjoint, written as $\mu_1 \perp \mu_2$ (while $\mu_1 + \mu_2 = \mu$). In other words, [18] assumed that μ_1, μ_2 must saturate μ on their respective supports. In addition, [18] assumed that μ is continuous, i.e., has no atoms. Under these assumptions, [18] showed that an optimal market model π for the problem (1.2) is given by the *left-curtain coupling* (see [8, 15, 18] for more details about this interesting martingale transport) along with an optimal exercise strategy B , and furthermore, the cheapest superhedge can be derived.

Now we would like to shift our focus and ask, “Under what conditions must the optimal option exercise be pure?” That is, when will an optimal μ_1 saturate μ , or equivalently, achieve $\mu_1 \perp \mu_2$? Note that the problem (1.2) can be rewritten as

$$(1.4) \quad P_c = \sup_{\mu_1 \leq \mu} P_c(\mu_1), \quad \text{where } P_c(\mu_1) := \sup_{\pi \in \mathcal{M}(\mu_1, \nu)} \mathbb{E}_{\gamma_1}[c_1] + \mathbb{E}_{\gamma_2}[c_2],$$

where $\gamma_l = \pi_x \otimes \mu_l$, $l = 1, 2$. Note that the problem (1.2) has a nonconvex domain in terms of the variable (γ_1, γ_2) . This is because even if $(\gamma_1, \gamma_2), (\gamma'_1, \gamma'_2)$ are feasible (i.e., sharing the same kernel respectively), the convex combination $(\frac{\gamma_1 + \gamma'_1}{2}, \frac{\gamma_2 + \gamma'_2}{2})$ may not share the same kernel thus infeasible, unless $\mu_1 = \mu'_1$ and $\mu_2 = \mu'_2$. On the other hand, the subproblem $P_c(\mu_1)$ has a convex domain in terms of (γ_1, γ_2) . This leads us to consider a relaxed problem (2.2) with its optimal value denoted by $\bar{P}_c$. Clearly $P_c \leq \bar{P}_c$; see Section 2 for details. Our result is the following.

Theorem 1.1. *Assume [A] and the cost form (1.1). Suppose $y \mapsto c_2(x, y)$ is strictly convex and $c_1(x) \neq c_2(x, x)$ for μ -a.e. x , and ν is absolutely continuous with respect to the Lebesgue measure. If $P_c = \bar{P}_c$, then every solution (π, μ_1) to the problem (1.4) satisfies $\mu_1 \perp \mu - \mu_1$. Furthermore, given any optimal candidate model π , the μ_1 yielding an optimal pair (π, μ_1) is unique.*

We note that the condition $c_1(x) > c_2(x, x)$ is natural because, if $c_1(x) \leq c_2(x, x)$ and c_2 is convex in y , it is always optimal to choose $c_2(x, y)$ by Jensen's inequality $c_2(x, x) \leq \int c_2(x, y) \pi_x(dy)$. Theorem 1.1 says that in this case, every optimal exercise, or stopping, is nonrandomized. Evidently, the problem (1.2) can be viewed as an optimal stopping problem, in which the option holder either stops at time 1 and receives the sure reward $c_1(x)$, or goes and receives the reward $c_2(x, y)$ (which is stochastic at time 1) at time 2. This naturally places the theorem in the context of the vast literature on the Skorokhod embedding problem [7, 16, 21], with the key difference that we now face uncertainty in the family of models $\mathcal{M}(\mu, \nu)$. Such model uncertainty was also considered in [2, 12] in continuous time setup. For more results on American options and their robust hedging, we refer to [4–6].

In the optimal transport literature, the absolute continuity of μ is typically assumed in order to derive non-randomizing solutions, known as Monge solutions. Continuity of μ was also assumed in [18]. In contrast, Theorem 1.1 assumes the absolute continuity of ν , while making no assumptions about μ . On the other hand, the equation assumption $P_c = \bar{P}_c$ imposed in the theorem appears to be highly restrictive, prompting us to seek a sufficient condition that yields the equation. For example, can the absolute continuity of μ with respect to Lebesgue measure imply the equation (with suitable additional conditions on the cost)?

Finally, the uniqueness of μ_1 given a fixed model π is obtained by a standard argument in optimal transport through mixing two optimal solutions and invoking the result $\mu_1 \perp \mu - \mu_1$. When (π, μ_1) and (π', μ'_1) are both optimal (with possibly $\pi \neq \pi'$), it is an open question whether $\mu_1 = \mu'_1$ under suitable conditions. This is due to the nonconvexity of the domain of the problem (1.2) in terms of (γ_1, γ_2) .

The remainder of the paper is structured as follows. The theorem will be proved utilizing a duality and its attainment result. They will be discussed in Section 2. Section 3 then presents proofs of the results.

2. DUALITY

In this section, we consider cost functions more general than (1.1), such as

$$(2.1) \quad \vec{c} = (c_1, c_2, \dots, c_L), \quad c_l = c_l(x, y) \in \mathbb{R}, \quad l = 1, 2, \dots, L.$$

Throughout this section, we assume the following.

[A'] c_l are continuous for all l , $\mu \preceq_c \nu$, and $\sum_{l=1}^L |c_l(x, y)| \leq v(x) + w(y)$ for some continuous functions $v \in L^1(\mu)$, $w \in L^1(\nu)$.

As noted, the domain of the problem (1.2), in terms of the variable (γ_1, γ_2) , is nonconvex. This leads us to consider a relaxed problem for (1.2); see also [1] for related results. Let $\mathcal{M} := \cup_{\mu \preceq_c \nu} \mathcal{M}(\mu, \nu)$, that is, $\mathcal{M}$ is the set of all martingale transports between some probability marginals in convex order, hence $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R}^2)$. Let $\overline{\mathcal{M}}$ be the set of all martingale transports with arbitrary nonnegative finite total mass, that is, $\gamma \in \overline{\mathcal{M}}$ if $\gamma \equiv 0$ or $\gamma/||\gamma|| \in \mathcal{M}$ where $||\gamma|| = \int_{\mathbb{R}^2} \gamma(dx, dy) \in (0, \infty)$ denotes the total mass. Define

$$\mathcal{M}_L(\mu, \nu) := \left\{ \vec{\gamma} = (\gamma_1, \dots, \gamma_L) \left| \sum_{l=1}^L \gamma_l \in \mathcal{M}(\mu, \nu) \text{ and } \gamma_l \in \overline{\mathcal{M}} \text{ for all } l = 1, \dots, L. \right. \right\}$$

$\mathcal{M}_L(\mu, \nu)$ is clearly convex. Now we define the relaxed problem

$$(2.2) \quad \overline{P}_c := \sup_{\vec{\gamma} \in \mathcal{M}_L(\mu, \nu)} \sum_{l=1}^L \mathbb{E}_{\gamma_l}[c_l].$$

The difference is that in (1.2) (with the generalized cost (2.1)), $\{\gamma_l\}_l$ are assumed to have the same kernel π_x inherited from a model $\pi \in \mathcal{M}(\mu, \nu)$, whereas in (2.2), this restriction is relaxed. Both problems satisfy the condition $\sum_l \gamma_l \in \mathcal{M}(\mu, \nu)$. Hence, $P_c \leq \overline{P}_c$.

We turn to the dual problem of (2.2). Define $\overline{\Psi}_c$ to be the space of functions $(\varphi, \psi, \vec{\theta}) = (\varphi, \psi, \theta_1, \dots, \theta_L)$ such that $\varphi \in C(\mathbb{R}) \cap L^1(\mu)$, $\psi \in C(\mathbb{R}) \cap L^1(\nu)$, $\theta_l \in C_b(\mathbb{R})$, satisfying

$$(2.3) \quad c_l(x, y) \leq \varphi(x) + \psi(y) + \theta_l(x)(y - x) \quad \text{for all } l = 1, \dots, L \text{ and } (x, y) \in \mathbb{R}^2.$$

The dual problem to (2.2) is now given by

$$(2.4) \quad \overline{D}_c := \inf_{(\varphi, \psi, \vec{\theta}) \in \overline{\Psi}_c} \mu(\varphi) + \nu(\psi).$$

A duality result is the following.

Proposition 2.1. *Assume [A']. Then $\overline{P}_c = \overline{D}_c$.*

For the financial meaning of the dual problems in terms of American option superhedging, we refer to [1, 3–6, 17, 18, 20]. The additional element required to prove Theorem 1.1 is the dual attainment result, which asserts that there is an appropriate solution to the dual problem (2.4). For $\xi \in \mathcal{P}(\mathbb{R})$, its potential function is defined by $u_\xi(x) := \int |x - y| d\xi(y)$. Then we say that a pair of probabilities (μ, ν) in convex order is irreducible if the set $I := \{x \in \mathbb{R} \mid u_\mu(x) < u_\nu(x)\}$ is a connected (open) interval containing the full mass of μ , i.e., $\mu(I) = \mu(\mathbb{R})$.

Proposition 2.2. *Assume [A'] and suppose (μ, ν) is irreducible. Then there exists a dual optimizer $(\varphi, \psi, \vec{\theta})$, $\varphi, \psi : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$, $\theta_l : \mathbb{R} \rightarrow \mathbb{R}$, that satisfies (2.3) tightly in the following pathwise sense (but needs not be in $\overline{\Psi}_c$):*

$$(2.5) \quad c_l(x, y) = \varphi(x) + \psi(y) + \theta_l(x)(y - x) \quad \gamma_l - \text{a.e.}, \quad \text{for all } l = 1, \dots, L$$

for every solution $\vec{\gamma} = (\gamma_1, \dots, \gamma_L)$ to the problem (2.2).

We emphasize that $(\varphi, \psi, \vec{\theta})$ may not be in $\overline{\Psi}_c$ but are only measurable, with φ, ψ real-valued μ, ν -a.s., respectively. They need not be integrable nor continuous.

3. PROOFS

Proof of Proposition 2.1. Let $\mathcal{N}$ be the set of all nonnegative finite measures on $\mathbb{R}^2$ (that do not need to be martingales.) For $\gamma \in \mathcal{N}$, let γ^X, γ^Y denote its marginal on the x, y -coordinate respectively. Let $\varphi \in C(\mathbb{R}) \cap L^1(\mu)$, $\psi \in C(\mathbb{R}) \cap L^1(\nu)$, $\theta_l \in C_b(\mathbb{R})$. We assert that the following equalities hold:

$$\begin{aligned} \overline{P}_c &= \sup_{\vec{\gamma} \in \mathcal{M}_L(\mu, \nu)} \sum_{l=1}^L \mathbb{E}_{\gamma_l}[c_l] \\ &= \sup_{\gamma_l \in \mathcal{N} \forall l} \inf_{(\varphi, \psi, \vec{\theta})} \sum_l \gamma_l(c_l) + (\mu - \sum_l \gamma_l^X)(\varphi) + (\nu - \sum_l \gamma_l^Y)(\psi) - \sum_l \gamma_l(\theta_l(x)(y - x)) \\ &= \inf_{(\varphi, \psi, \vec{\theta})} \sup_{\gamma_l \in \mathcal{N} \forall l} \mu(\varphi) + \nu(\psi) + \sum_l \gamma_l(c_l(x, y) - \varphi(x) - \psi(y) - \theta_l(x)(y - x)) \\ &= \inf_{c_l(x, y) \leq \varphi(x) + \psi(y) + \theta_l(x)(y - x) \forall l} \mu(\varphi) + \nu(\psi) = \overline{D}_c. \end{aligned}$$

The derivation of the equalities is fairly standard: the second equality holds because the infimum achieves $-\infty$ as soon as $\sum_l \gamma_l^X \neq \mu$, $\sum_l \gamma_l^Y \neq \nu$, or $\gamma_l \notin \overline{\mathcal{M}}$, implying that $\vec{\gamma}$ in the second line must be in $\mathcal{M}_L(\mu, \nu)$ to achieve the first supremum. The

third equality is based on a standard minimax theorem, which asserts that the equality holds when the sup and inf are swapped. Because the objective function is bilinear, i.e., linear in each variable $((\gamma_l)_l \text{ and } (\varphi, \psi, \vec{\theta}))$, the minimax theorem holds in this case and we omit the detail. The fourth equality is because, if $c_l(x, y) - \varphi(x) - \psi(y) - \theta_l(x)(y - x) > 0$ for some $(x, y) \in \mathbb{R}^2$, one can select $\gamma_l \in \mathcal{N}$ such that the last supremum in the third line achieves $+\infty$, which hinders to achieve the first infimum. This implies $c_l(x, y) - \varphi(x) - \psi(y) - \theta_l(x)(y - x) \leq 0$ for all (x, y) , in which case it is best to choose $\gamma_l \equiv 0$ for the supremum in the third line. $\square$

Proof of Proposition 2.2. The proof consists of extending the ideas in [8, 9] to the vectorial cost (2.1). We will follow the five steps illustrated in [19], thereby omitting some details here but referring to the corresponding steps in [19].

Step 1. $\sum_{l=1}^L |c_l(x, y)| \leq v(x) + w(y)$ for some continuous functions $v \in L^1(\mu)$, $w \in L^1(\nu)$. A dual optimizer exists for $\vec{c}$ iff so does for $\tilde{c} := (c_l(x, y) + v(x) + w(y))_l$. Thus by replacing $\vec{c} = (c_1, \dots, c_L)$ with $\tilde{c}$, from now on we assume $c_l \geq 0$ for all l .

As $\bar{P}_c = \bar{D}_c \in \mathbb{R}$, we can find an approximating dual optimizer $(\varphi_n, \psi_n, \theta_{l,n}) \in \bar{\Psi}_c$, $n \in \mathbb{N}$, such that the following duality holds (for all $l = 1, \dots, L$):

$$(3.1) \quad \varphi_n(x) + \psi_n(y) + \theta_{l,n}(x)(y - x) \geq c_l(x, y) \geq 0,$$

$$(3.2) \quad \mu(\varphi_n) + \nu(\psi_n) \searrow \bar{P}_c \text{ as } n \rightarrow \infty.$$

Define $f_n = -\varphi_n$, $h_{l,n} = -\theta_{l,n}$, so that (3.1) becomes

$$(3.3) \quad f_n(x) + h_{l,n}(x)(y - x) \leq \psi_n(y) - c_l(x, y) \leq \psi_n(y).$$

Define the convex functions

$$(3.4) \quad \chi_{l,n}(y) := \sup_{x \in \mathbb{R}} f_n(x) + h_{l,n}(x)(y - x), \quad \chi_n := \sup_{l=1, \dots, L} \chi_{l,n}.$$

Notice $\chi_{l,n}(y) \geq f_n(y) + h_{l,n}(y)(y - y) = f_n(y)$ for all $y \in \mathbb{R}$. Hence,

$$(3.5) \quad f_n \leq \chi_n \leq \psi_n \text{ for all } n.$$

By (3.2), this yields the uniform integral bound

$$(3.6) \quad \int \chi_n d(\nu - \mu) \leq \nu(\psi_n) - \mu(f_n) \leq C \text{ for all } l = 1, \dots, L \text{ and } n \in \mathbb{N}.$$

Using (3.6) and the assumption that (μ, ν) is irreducible, a local uniform boundedness of $\{\chi_n\}_n$ can be obtained (cf. Step 1 in the proof of [19, Theorem 1.2]):

there exists an increasing sequence of compact intervals $J_k := [c_k, d_k]$ and constants $M_k \geq 0$ for each $k \in \mathbb{N}$, such that $\cup_{k=1}^{\infty} J_k = J$, and

$$(3.7) \quad 0 \leq \sup_n \chi_n \leq M_k \text{ in } J_k.$$

Step 2. Given any approximating dual optimizer $(\varphi_n, \psi_n, \theta_{l,n})$ satisfying (3.2), (3.3), the goal is to suitably modify it and deduce pointwise convergence of φ_n, ψ_n to some functions φ, ψ μ, ν -a.s. as $n \rightarrow \infty$, respectively, where $\varphi, \psi \in \mathbb{R} \cup \{+\infty\}$ is μ, ν -a.s. finite. From convexity of χ_n with $\mu \preceq_c \nu$, we deduce, for all n ,

$$(3.8) \quad C \geq \nu(\psi_n) - \mu(f_n) \geq \nu(\chi_n) - \mu(f_n) \geq \mu(\chi_n) - \mu(f_n) = \|\chi_n - f_n\|_{L^1(\mu)},$$

Meanwhile, (3.3) gives $f_n(x) + h_{l,n}(x)(y - x) - \psi_n(y) \leq -c_l(x, y) \leq 0$, hence

$$f_n(x) + h_{l,n}(x)(y - x) - \psi_n(y) \leq \chi_n(y) - \psi_n(y) \leq 0.$$

Integrating by any $\pi \in \mathcal{M}(\mu, \nu)$ implies

$$(3.9) \quad \|\psi_n - \chi_n\|_{L^1(\nu)} \leq \nu(\psi_n) - \mu(f_n) \leq C \text{ for all } n.$$

These uniform L^1 bounds, combined with the local uniform bound (3.7) and Komlós compactness theorem, can imply the desired almost sure convergence of $\{\varphi_n\}$ and $\{\psi_n\}$ as presented in [9] and in Step 2 in the proof of [19, Theorem 1.2], thus we omit the detail here. Also, by following Step 3 in the same proof, one can deduce the following pointwise convergence of χ_n to a convex function χ

$$(3.10) \quad \lim_{n \rightarrow \infty} \chi_n(y) = \chi(y) \in \mathbb{R} \text{ for every } y \in J.$$

Step 3. We have obtained the almost sure limit functions φ, ψ , with $f := -\varphi$. We may define $\varphi := +\infty$ on a μ -null set which includes $\mathbb{R} \setminus I$, and $\psi := +\infty$ on a ν -null set which includes $\mathbb{R} \setminus J$, so that they are defined everywhere on $\mathbb{R}$. We will show there exists a function $\theta_l : \mathbb{R} \rightarrow \mathbb{R}$, with $h_l := -\theta_l$, $l = 1, \dots, L$, such that

$$(3.11) \quad \varphi(x) + \psi(y) + \theta_l(x)(y - x) \geq c_l(x, y).$$

For any function $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ which is bounded below by an affine function, let $\text{conv}[f] : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ denote the lower semi-continuous convex envelope of f , that is the supremum of all affine functions λ satisfying $\lambda \leq f$ (If there is no

such λ , let $\text{conv}[f] \equiv -\infty$.) Set $H_{l,n}(x, y) := \text{conv}[\psi_n(\cdot) - c_l(x, \cdot)](y)$. By (3.3),

$$(3.12) \quad f_n(x) + h_{l,n}(x)(y - x) \leq H_{l,n}(x, y) \leq \psi_n(y) - c_l(x, y),$$

because the left hand side is affine in y . Letting $y = x$ gives $f_n(x) \leq H_{l,n}(x, x)$.

Next, since the lim sup of convex functions is convex, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} H_{l,n}(x, y) &\leq \text{conv}[\limsup_{n \rightarrow \infty} (\psi_n(\cdot) - c_l(x, \cdot))](y) \\ &\leq \text{conv}[\psi(\cdot) - c_l(x, \cdot)](y) =: H_l(x, y). \end{aligned}$$

Then by the convergence $f_n \rightarrow f$ and the definition of $H_l(x, y)$, we get

$$f(x) \leq H_l(x, x), \text{ and } H_l(x, y) \leq \psi(y) - c_l(x, y).$$

Set $A := \{x \in I \mid \lim_{n \rightarrow \infty} f_n(x) = f(x) \in \mathbb{R}\}$, so that $\mu(A) = 1$. Since $y \mapsto H_l(x, y)$ is continuous in J for every $x \in A$ due to the convexity of $y \mapsto H_l(x, y)$ and ν -a.s. finiteness of ψ , the subdifferential $\partial H_l(x, \cdot)(y)$ is nonempty, convex and compact for every $y \in I = \text{int}(J)$. This allows us to choose a measurable function $h_l : A \rightarrow \mathbb{R}$ satisfying $h_l(x) \in \partial H_l(x, \cdot)(x)$. Such choice yields (3.11) as follows:

$$f(x) + h_l(x)(y - x) \leq H_l(x, x) + h_l(x)(y - x) \leq H_l(x, y) \leq \psi(y) - c_l(x, y).$$

We may define $h_l \equiv 0$ on $\mathbb{R} \setminus A$, noting that $f := -\infty$ on $\mathbb{R} \setminus A$.

Step 4. We will show that for any functions $\theta_l : \mathbb{R} \rightarrow \mathbb{R}$, $l = 1, \dots, L$ that satisfies (3.11) (whose existence was shown in the previous step), and for any maximizer $\vec{\gamma}^* = (\gamma_1^*, \dots, \gamma_L^*) \in \mathcal{M}_L(\mu, \nu)$ for the problem (2.2), it holds

$$(3.13) \quad \varphi(x) + \psi(y) + \theta_l(x)(y - x) = c_l(x, y) \quad \gamma_l^* - a.e. \text{ for all } l = 1, \dots, L.$$

For any $\vec{\gamma} = (\gamma_1, \dots, \gamma_L) \in \mathcal{M}_L(\mu, \nu)$, Assumption **[A]** yields $c_l \in L^1(\gamma_l)$. We claim

$$(3.14) \quad \begin{aligned} \liminf_{n \rightarrow \infty} \sum_{l=1}^L \int (\varphi_n(x) + \psi_n(y) + \theta_{l,n}(x)(y - x)) d\gamma_l \\ \geq \sum_{l=1}^L \int (\varphi(x) + \psi(y) + \theta_l(x)(y - x)) d\gamma_l \quad \text{for every } l. \end{aligned}$$

To see how the claim implies (3.13), let $\vec{\gamma}^*$ be any maximizer for (2.2). Then

$$\begin{aligned}
\bar{P}_c &= \lim_{n \rightarrow \infty} \sum_{l=1}^L \int (\varphi_n(x) + \psi_n(y) + \theta_{l,n}(x)(y-x)) d\gamma_l^* \\
&\geq \sum_{l=1}^L \liminf_{n \rightarrow \infty} \int (\varphi_n(x) + \psi_n(y) + \theta_{l,n}(x)(y-x)) d\gamma_l^* \\
&\geq \sum_{l=1}^L \int (\varphi(x) + \psi(y) + \theta_l(x)(y-x)) d\gamma_l^* \\
&\geq \sum_{l=1}^L \int c_l(x, y) d\gamma_l^* = \bar{P}_c,
\end{aligned}$$

hence equality holds throughout. Notice this yields (3.13), hence the proposition.

To prove (3.14), fix any $\vec{\gamma} = (\gamma_1, \dots, \gamma_L) \in \mathcal{M}_L(\mu, \nu)$. The nonnegativity (3.1) gives $\gamma_l^X(\varphi_n) + \gamma_l^Y(\psi_n) \geq 0$, and (3.2) gives $\sum_{l=1}^L (\gamma_l^X(\varphi_n) + \gamma_l^Y(\psi_n)) = \mu(\varphi_n) + \nu(\psi_n) \searrow \bar{P}_c$. This implies the sequence $\{\gamma_l^X(\varphi_n) + \gamma_l^Y(\psi_n)\}_n$ is bounded for all l . With this and (3.5), as in Step 2 (but $\gamma_l^X \preceq_c \gamma_l^Y$ instead of $\mu \preceq_c \nu$), we deduce

$$\sup_n \|\chi_n + \varphi_n\|_{L^1(\gamma_l^X)} < \infty, \quad \sup_n \|\psi_n - \chi_n\|_{L^1(\gamma_l^Y)} < \infty, \quad \text{for all } l.$$

From this, since $\varphi_n \rightarrow \varphi$, $\psi_n \rightarrow \psi$, $\chi_n \rightarrow \chi$, by Fatou's lemma, we get

$$\chi + \varphi \in L^1(\gamma_l^X), \quad \psi - \chi \in L^1(\gamma_l^Y),$$

$$\liminf_{n \rightarrow \infty} \int (\chi_n + \varphi_n) d\gamma_l^X \geq \int (\chi + \varphi) d\gamma_l^X, \quad \liminf_{n \rightarrow \infty} \int (\psi_n - \chi_n) d\gamma_l^Y \geq \int (\psi - \chi) d\gamma_l^Y.$$

This allows us to proceed

$$\begin{aligned}
&\liminf_{n \rightarrow \infty} \int (\varphi_n(x) + \psi_n(y) + \theta_{l,n}(x)(y-x)) d\gamma_l \\
&= \liminf_{n \rightarrow \infty} \int (\varphi_n(x) + \chi_n(x) - \chi_n(y) + \psi_n(y) - \chi_n(x) + \chi_n(y) + \theta_{l,n}(x)(y-x)) d\gamma_l \\
&\geq \int (\chi + \varphi) d\gamma_l^X + \int (\psi - \chi) d\gamma_l^Y + \liminf_{n \rightarrow \infty} \int (\chi_n(y) - \chi_n(x) + \theta_{l,n}(x)(y-x)) d\gamma_l.
\end{aligned}$$

To handle the last term, disintegrate $\gamma_l = (\gamma_l)_x \otimes \gamma_l^X$, and let $\xi_n : I \rightarrow \mathbb{R}$ be a sequence of functions satisfying $\xi_n(x) \in \partial\chi_n(x)$. This allows us to proceed

$$\begin{aligned} & \int (\chi_n(y) - \chi_n(x) + \theta_{l,n}(x)(y - x)) d\gamma_l \\ &= \iint (\chi_n(y) - \chi_n(x) + \theta_{l,n}(x)(y - x)) (\gamma_l)_x(dy) \gamma_l^X(dx) \\ &= \iint (\chi_n(y) - \chi_n(x) + \xi_n(x)(y - x)) (\gamma_l)_x(dy) \gamma_l^X(dx), \end{aligned}$$

because $\int \theta_{l,n}(x)(y - x) (\gamma_l)_x(dy) = \int \xi_n(x)(y - x) (\gamma_l)_x(dy) = 0$. Notice that the last integrand is nonnegative. Thus by repeated Fatou's lemma, we deduce

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int (\chi_n(y) - \chi_n(x) + \theta_{l,n}(x)(y - x)) d\gamma_l \\ & \geq \int \liminf_{n \rightarrow \infty} \left(\int (\chi_n(y) - \chi_n(x) + \xi_n(x)(y - x)) (\gamma_l)_x(dy) \right) \gamma_l^X(dx) \\ & \geq \int \left(\int (\chi(y) - \chi(x) + \xi(x)(y - x)) (\gamma_l)_x(dy) \right) \gamma_l^X(dx), \end{aligned}$$

for some $\xi(x) \in \partial\chi(x)$ which is a limit point of the bounded sequence $\{\xi_n(x)\}_n$. Finally, in the last line, the inner integral equals

$$\int (\chi(y) - \chi(x) + \theta_l(x)(y - x)) (\gamma_l)_x(dy).$$

This proves the claim, hence the proposition. $\square$

We are prepared to prove Theorem 1.1.

Proof of Theorem 1.1. Fix any optimal pair (π, μ_1) for the problem (1.2), and let $\gamma_l = \pi_x \otimes \mu_l$, $l = 1, 2$, with $\mu_2 = \mu - \mu_1$ and the kernel $\{\pi_x\}_x$ inherited from π . We understand $c_1(x, y) = c_1(x)$ in the proof. Let us first assume that $\mu \preceq_c \nu$ is irreducible. Because we assume $P_c = \bar{P}_c$, by Proposition 2.2, with $f = -\varphi$ and $h_l = -\theta_l$, we have

$$(3.15) \quad f(x) + h_l(x)(y - x) + c_l(x, y) \leq \psi(y) \quad \text{for each } l = 1, 2 \text{ and } (x, y) \in \mathbb{R}^2,$$

$$(3.16) \quad f(x) + h_l(x)(y - x) + c_l(x, y) = \psi(y) \quad \gamma_l - a.e. (x, y) \text{ for each } l = 1, 2.$$

Now, saying that an American option holder randomizes her exercise between c_1, c_2 is equivalent to saying that the common mass of μ_1, μ_2 (written as $\mu_1 \wedge \mu_2$) is nonzero. The common mass of μ_1, μ_2 is defined by the largest measure $\rho = \mu_1 \wedge \mu_2$

satisfying $\rho \leq \mu_1$ and $\rho \leq \mu_2$. Since γ_1 and γ_2 have the same kernel, (3.16) implies

$$(3.17) \quad f(x) + h_l(x)(y - x) + c_l(x, y) = \psi(y) \quad \pi_x \otimes \rho - a.e. (x, y) \text{ for } l = 1, 2.$$

Observe that ψ can be taken as $\psi := \max(\psi_1, \psi_2)$, where

$$\psi_l(y) := \sup_x f(x) + h_l(x)(y - x) + c_l(x, y),$$

and consequently, ψ_1, ψ_2, ψ are all convex since c_2 is convex in y (while c_1 is independent of y .) Now the idea is to differentiate (3.17) by y for ν -a.e. y , which is enabled by the fact that ψ is differentiable ν -a.s., since ν is assumed to be absolutely continuous with respect to Lebesgue. By the differentiation combined with the first-order optimality condition from (3.15), (3.16) for each $l = 1, 2$, we deduce

$$(3.18) \quad h_1(x) = \psi'(y) = h_2(x) + (c_2)_y(x, y) \quad \pi_x \otimes \rho - a.e. (x, y),$$

where $(c_2)_y$ denotes the partial derivative of c_2 by y , noting that (3.15), (3.16) implies $(c_2)_y(x, y)$ exists γ_2 -a.e., since ψ is differentiable ν -a.e..

Now since $c_1 = c_1(x)$, the left hand side of (3.15) is linear in y when $l = 1$, while ψ is convex. With this, the first equality in (3.18) implies that for ρ -a.e. x , ψ is linear in the smallest interval containing $\text{spt}(\pi_x)$ which contains x . Hence,

$$(3.19) \quad \psi'(y) = \psi'(x) \quad \pi_x \otimes \rho - a.e. (x, y).$$

The second equality in (3.18) thus becomes

$$(3.20) \quad (c_2)_y(x, y) = \psi'(x) - h_2(x) \quad \pi_x \otimes \rho - a.e. (x, y).$$

Because c_2 is assumed to be strictly convex in y , the solution y to (3.20) must be unique, and hence, $y = x$ since π_x has its barycenter at x . We conclude

$$(3.21) \quad \pi_x = \delta_x \quad \rho - a.e. x,$$

where $\delta_x \in \mathcal{P}(\mathbb{R})$ is the Dirac mass at x . (3.17) then yields

$$(3.22) \quad c_1(x) = c_2(x, x) \quad \rho - a.e. x.$$

Now if $c_1(x) \neq c_2(x, x)$ μ -a.s., then (3.22) implies $\rho \equiv 0$, yielding $\mu_1 \perp \mu - \mu_1$ for any optimal pair (π, μ_1) . This proves the disjointness when $\mu \preceq_c \nu$ is irreducible.

For general $\mu \preceq_c \nu$, it is well known that any convex-ordered pair (μ, ν) can be decomposed as at most countably many irreducible pairs, and the decomposition

is uniquely determined by the potential functions u_μ, u_ν . More precisely, we have: [9, Proposition 2.3] Let $(I_k)_{1 \leq k \leq N}$ be the open components of the open set $\{u_\mu < u_\nu\}$ in $\mathbb{R}$, where $N \in \mathbb{N} \cup \{+\infty\}$. Let $I_0 = \mathbb{R} \setminus \cup_{k \geq 1} I_k$ and $\mu_k = \mu|_{I_k}$ for $k \geq 0$, so that $\mu = \sum_{k \geq 0} \mu_k$. There exists a unique decomposition $\nu = \sum_{k \geq 0} \nu_k$ such that

$$\mu_0 = \nu_0, \text{ and } (\mu_k, \nu_k) \text{ is irreducible for } k \geq 1 \text{ with } \mu_k(I_k) = \mu_k(\mathbb{R}).$$

Moreover, any $\pi \in \mathcal{M}(\mu, \nu)$ admits a unique decomposition $\pi = \sum_{k \geq 0} \pi_k$ such that $\pi_k \in \mathcal{M}(\mu_k, \nu_k)$ for all $k \geq 0$.

Here, π_0 must be the identity transport, i.e., $(\pi_0)_x = \delta_x$, since it is a martingale transport between the same marginal. Since the theorem has already been proven for the irreducible pairs (μ_k, ν_k) , $k \geq 1$, we only need to prove it for the identity transport π_0 . In this case, $\int c_2(x, y)(\pi_0)_x(dy) = c_2(x, x)$, yielding that it is optimal to exercise c_1 when $c_1(x) > c_2(x, x)$, while it is optimal to exercise c_2 when $c_1(x) < c_2(x, x)$. The assumption $c_1(x) \neq c_2(x, x)$ μ -a.s. therefore proves $\mu_1 \perp \mu - \mu_1$.

Finally, if (π, μ_1) and (π, μ'_1) are both optimal, let $\gamma_l = \pi_x \otimes \mu_l$ and $\gamma'_l = \pi_x \otimes \mu'_l$, $l = 1, 2$. Let $\tilde{\gamma}_l = (\gamma_l + \gamma'_l)/2$. Then $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ is an optimal solution to (1.2) since γ_l and γ'_l share the same kernel. Now $\mu_1 \neq \mu'_1$ implies $\tilde{\gamma}_1^X \not\perp \tilde{\gamma}_2^X$, a contradiction. $\square$

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